

Factorials Stretch

11. $\frac{7!}{6!} = \frac{7 \times \cancel{6!}}{\cancel{6!}} = 7$

12. $\frac{4!+5!}{4!} = \frac{4!+5(4!)}{4!} = \frac{4!(1+5)}{4!} = 1+5 = 6$

13. $\frac{4!+5!}{6} = \frac{4!+5(4!)}{6} = \frac{4!(1+5)}{6} = \frac{4!(6)}{6} = 4! = 4 \times 3 \times 2 \times 1 = 24$

14. $n! = 10 \times 4 \times 3 = (2 \times 5) \times (2 \times 2) \times 3 = 5 \times 4 \times 3 \times 2 = 5!$. So, $n = 5$.

15. The inequality $(n+1)! - n! < n^3$ can be rewritten as $(n+1)(n!) - n! < n^3 \rightarrow n!(n+1-1) < n^3 \rightarrow n!(n) < n^3 \rightarrow n! < n^2$. So, we are looking for the greatest positive integer n such that the square of n is greater than $n!$. Starting with $n = 2$, we see that $2! = 2$ and $2^2 = 4$. So, $n! < n^2$ for $n = 2$. Next, for $n = 3$, we see that $3! = 6$ and $3^2 = 9$. So, $n! < n^2$ for $n = 3$. Next, we try $n = 4$ and get $4! = 24$ and $4^2 = 16$. So, $n! > n^2$ for $n = 4$. For $n > 4$, $(n-1)(n-2) > n$, so $n! > n(n-1)(n-2) > n^2$. Therefore, the greatest positive integer value that satisfies the original inequality is **3**.

16. We are looking for the greatest n such that $n!$ has at most 6 factors of $10 = 2 \times 5$. For example, if $n = 15$, then $15! = 15 \times 14 \times \cdots \times 2 \times 1$. When we consider the prime factorization of each integer being multiplied, the integers 2 and 5 can be multiplied to get a factor of 10. Then the integer 10 itself counts as another. Finally, since $15 = 3 \times 5$, when this 5 is multiplied by, say, a 2 from, say, $4 = 2^2$, we get another factor of 10. So, the greatest power of 10 that $15!$ is divisible by is 10^3 . Since there are more 2s than 5s among the factors, we will keep track of the 5s among those factors. To be divisible by 10^6 , then, the desired factorial must multiply 5, 10, 15, 20 and 25 (which contains two factors of 5), but not 30 because that would give us 7 factors of 10. The greatest value of n less than 30 is **29**.

17. As with problem 16, we need to count the number of factors of 10 in $125!$ to determine the number of trailing zeros. Since there are more 2s than 5s among the factors, we will keep track of the 5s among those factors. From 5 through 125, there are 25 multiples of 5, each contributing at least one factor of 5. Also, 25, 50, 75, 100 and 125 each contribute another factor of 5, since they are all multiples of $25 = 5^2$, and 125 contributes still another factor of 5, since $125 = 5^3$. That's a total of $25 + 5 + 1 = 31$ factors of 5, and thus 31 factors of 10. Therefore, $125!$ is divisible by 10^{31} but not by 10^{32} , and the number of trailing zeros is **31** zeros.

18. $\frac{11!-10!}{6!-5!} = \frac{11(10!)-10!}{6(5!)-5!} = \frac{10!(11-1)}{5!(6-1)} = \frac{10!(10)}{5!(5)} = \frac{10!(2)}{5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times (\cancel{5!})(2)}{\cancel{5!}} = 10 \times 9 \times 8 \times 7 \times 6 \times 2$. Rewriting this expression as the product of prime numbers yields $(2 \times 5) \times 3^2 \times 2^3 \times 7 \times (2 \times 3) \times 2$, which shows that the greatest prime divisor of the original expression is **7**.

19. Expanding $4! \times 3! \times 2! \times 1!$, we get $(4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1) \times (2 \times 1) \times 1$. The prime factorization of this quantity is $2^5 \times 3^2$. A perfect square divisor of the quantity must have either 2 or 3 or both as factors an even number of times. The number 2 can be a factor of such a divisor 0, 2 or 4 times, and 3 can be a factor 0 or 2 times. The number of divisors of $4! \times 3! \times 2! \times 1!$ that are perfect squares is $3 \times 2 = 6$ divisors. They are 1^2 , 2^2 , 3^2 , $2^4 = 4^2$, $2^2 \times 3^2 = 6^2$ and $2^4 \times 3^2 = 12^2$.

20. There are 25 positive integers $n \leq 25$. There is no way for n^2 to be a factor of $n!$ for a prime number since there is no other multiple of n being multiplied in $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. So, we can exclude the 9 prime numbers among those 25 positive integers. That leaves us with $25 - 9 = 16$ integers. We also need to exclude 4 since $4^2 = 16$, which is not a factor of $4! = 4 \times 3 \times 2 \times 1$. It can be verified that in all other cases, there are enough smaller factors of n present that, when multiplied together, produce another factor of n in $n!$. So, the number of positive integers n that meet the given criteria is $16 - 1 = 15$ integers.