

Divisor Counting Stretch

1. Since $6000 = 2^4 \times 3^1 \times 5^3$, the number of divisors of 6000 is $(4 + 1)(1 + 1)(3 + 1) = 5 \times 2 \times 4 = 40$ divisors.
2. Since the 6000 has 40 divisors, we seek the smallest positive integer with 40 divisors. Because $40 = 5 \times 2 \times 2 \times 2$, an integer with 40 divisors can have prime factorization $p^4 \times q^1 \times r^1 \times s^1$. To make the integer as small as possible, we assign the largest exponent to the smallest prime and use the smallest possible distinct primes, giving $2^4 \times 3^1 \times 5^1 \times 7^1$. Any other factorization of 40 uses fewer prime factors and therefore requires larger exponents, producing a larger integer. Thus, the smallest positive integer with 40 divisors is $2^4 \times 3 \times 5 \times 7 = 16 \times 105 = 1680$.
3. Since 13 is prime, an integer with exactly 13 divisors must have the form p^{12} , where p is prime. The smallest such number is $2^{12} = 4096$.
4. Since $100 = 5 \times 5 \times 2 \times 2$, one possible exponent pattern for a integer with 100 divisors is 4, 4, 1, 1. To make the integer as small as possible, we assign the largest exponents to the smallest primes and use the smallest possible distinct primes, giving prime factorization $2^4 \times 3^4 \times 5^1 \times 7^1$. Any other factorization of 100 uses fewer prime factors and therefore requires larger exponents, producing a larger integer. Thus the smallest positive integer with 100 divisors is $2^4 \times 3^4 \times 5 \times 7 = 16 \times 81 \times 35 = 45,360$.
5. Since $450 = 2^1 \times 3^2 \times 5^2$, an odd divisor cannot contain the factor 2. Therefore, there are $(2 + 1)(2 + 1) = 3 \times 3 = 9$ odd divisors.
6. Since $7200 = 2^5 \times 3^2 \times 5^2$, it has $(5 + 1)(2 + 1)(2 + 1) = 54$ divisors. A divisor is a multiple of 6 if it contains at least one factor of 2 and at least one factor of 3. There are $5 \times 2 \times 3 = 30$ such divisors. Therefore, the probability is $30/54 = 5/9$.
7. We want the least positive integer n that has 12 divisors, 6 of which, or $1/2$, are even. That means 2^1 divides n but 2^2 does not. Similarly, 6 out of 12 of the divisors, or $1/2$, are to be multiples of 3, so 3^1 divides n but 3^2 does not. With 2^1 and 3^1 , n would have $(1 + 1)(1 + 1) = 4$ divisors. Since we need 12 divisors, we must multiply the divisor count by 3, and the remaining factor in the prime factorization must be of the form p^2 . To make n as small as possible, we need p to be the smallest prime that we have not used, thus 5. Therefore, $n = 2^1 \times 3^1 \times 5^2 = 150$.
8. Since $72 = 2^3 \times 3^2$, it has $(3 + 1)(2 + 1) = 12$ divisors. Each divisor pairs with a complementary divisor that multiplies with it to give 72, so the divisors form 6 pairs. Therefore, the product of all divisors of 72 is 72^6 . Since 72^6 has the same units digit as 2^6 , and $2^6 = 64$, the units digit is 4.
9. An integer has exactly 10 divisors if its prime factorization is either p^9 or p^4q , where p and q are distinct primes. The smallest such integer is $2^4 \times 3 = 48$, and the next smallest is $2^4 \times 5 = 80$. Their positive difference is $80 - 48 = 32$.
10. Since $126,000 = 2^4 \times 3^2 \times 5^3 \times 7^1$, it has $(4 + 1)(2 + 1)(3 + 1)(1 + 1) = 120$ divisors. A divisor is a multiple of 20 if it contains at least two factors of 2 and at least one factor of 5. There are $3 \times 3 \times 3 \times 2 = 54$ such divisors. Therefore, the probability is $54/120 = 9/20$.