

Answer Sheet

Number	Answer	Problem ID
1	87 points	22103
2	22500	DC444
3	$\frac{1}{2}$	22454
4	4	41B4D
5	$-\frac{5}{3}$	D0403
6	30	34A4D
7	0 integers	A01D4
8	250	3DC53
9	11	20214
10	1.5 inches	03544
11	30 degrees	C5A54
12	-6	1C544
13	289 people	14AD3
14	44	A0403
15	4 cents	1C
16	24/13 hours	DBB54
17	15 units	01003
18	961	CA0D4
19	28 %	C35D1
20	37	DA424

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Solutions

- (1) **87 points** ID: [22103]

Set the average for her previous exams to be x . The total amount of points including the final will be $6x + 2 \cdot 99$. The average is $\frac{6x + 2 \cdot 99}{8} = 90$. Then we solve for x .

$$\frac{6x + 2 \cdot 99}{8} = 90 \rightarrow 6x + 198 = 720 \rightarrow 6x = 522 \rightarrow x = \boxed{87}.$$

- (2) **22500** ID: [DC444]

The sum of the first 150 odd numbers can be written as $1 + 3 + 5 + 7 + \cdots + 299$, or

$$(2 - 1) + (4 - 1) + (6 - 1) + (8 - 1) + \cdots + (300 - 1)$$

Rewriting it again, this sum is equivalent to $-150 + (2 + 4 + 6 + 8 + \cdots + 300)$. Finally, we can factor out a two from the sum, leaving us with

$$-150 + 2(1 + 2 + 3 + 4 + \cdots + 150)$$

The sum of the first n positive integers is $\frac{n(n+1)}{2}$, so the sum inside the parentheses is

$$1 + 2 + 3 + 4 + \cdots + 150 = \frac{150(151)}{2} = 11325$$

Plugging this into the rest of the expression,

$$-150 + 2(11325) = \boxed{22500}$$

- (3) **1/2** ID: [22454]

The width is $\frac{21}{2} - 7 = \frac{7}{2}$, so the ratio of width to length is $\boxed{\frac{1}{2}}$.

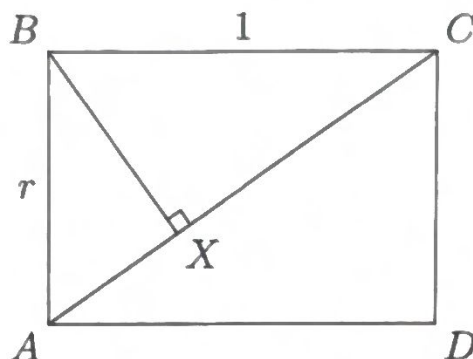
(4) **4** ID: [41B4D]

Rectangle $ABCD$ is partitioned into three right triangles as shown: $\triangle ABX$ (smallest), $\triangle BCX$ (middle-sized), and $\triangle CAD$ (largest), with all three similar to one another. Since the areas of the three triangles form an arithmetic progression, let the areas in increasing order be $[\triangle ABX] = x - d$, $[\triangle BCX] = x$, and $[\triangle CAD] = x + d$.

We know that $(x - d) + x = (x + d)$, so $2x - d = x + d$, and $x = 2d$. So, the triangles have areas $[\triangle ABX] = d$, $[\triangle BCX] = 2d$, and $[\triangle CAD] = 3d$.

That means $\triangle ABX$ and $\triangle BCX$ have areas in the ratio $\frac{[\triangle ABX]}{[\triangle BCX]} = \frac{1}{2}$, so corresponding sides AB and BC

must have sides in the ratio $\frac{AB}{BC} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, so $a = 2$, $b = 2$, and $ab = \boxed{4}$.



(5) $-\frac{5}{3}$ ID: [D0403]

Since $h(x) = f(x) + g(x)$, we have that $h(x) = (5x - 3) + (3x^2 + 1) = 3x^2 + 5x - 2$. We want to find all x such that $h(x) = 0$, so we set the expression $3x^2 + 5x - 2$ equal to 0 to get the quadratic equation $3x^2 + 5x - 2 = 0$. Factoring this quadratic, we have that

$$\begin{aligned} 3x^2 + 6x - x - 2 &= 0 \\ \Rightarrow (3x - 1)(x + 2) &= 0 \end{aligned}$$

Thus, the two x -values for which $h(x) = 0$ are $x = \frac{1}{3}$ and $x = -2$ and their sum is $\frac{1}{3} + (-2) = \boxed{-\frac{5}{3}}$.

(6) **30** ID: [34A4D]

The number of marbles Jon and Jeremy have combined is $240 \div 3 = 80$. Since 50 of those are Jon's, that leaves $80 - 50 = \boxed{30 \text{ marbles}}$ for Jeremy.

(7) **0 integers** ID: [A01D4]

Since the absolute value of anything is always positive, there are $\boxed{0}$ such integers x that could satisfy this impossible inequality.

- (8) **250** ID: [3DC53]

The positive integer is equal to $35 \cdot 7 + 5 = (30 + 5) \cdot 7 + 5 = 210 + 35 + 5 = \boxed{250}$.

- (9) **11** ID: [20214]

The third side will be shortest if it is one of the legs, the hypotenuse is the side of length 12, and the other leg is the side with 5. By the Pythagorean theorem, the third side then has length $\sqrt{144 - 25} = \sqrt{119} \approx \sqrt{121} = \boxed{11}$.

- (10) **1.5 inches** ID: [03544]

The border has some width w , so the entire frame must have dimensions $9 + 2w$ by $12 + 2w$ inches. Since the area is 180 square inches, we have

$$(9 + 2w)(12 + 2w) = 180.$$

Expanding the left-hand side gives

$$4w^2 + 42w + 108 = 180,$$

and dividing by 2 and rearranging gives

$$2w^2 + 21w - 36 = 0.$$

Factoring gives

$$(2w - 3)(w + 12) = 0.$$

Since w must be positive, we know that we have $2w - 3 = 0$, indicating that $w = \boxed{1.5}$ inches.

- (11) **30 degrees** ID: [C5A54]

Since the angles of a triangle sum to 180° , we can set up an equation to solve for x .

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{6} = 180^\circ$$

Multiplying both sides by 6 to get rid of the fractions,

$$3x + 2x + x = 1080^\circ$$

$$6x = 1080^\circ$$

$$x = 180^\circ$$

Thus, $\frac{x}{6}$ represents the smallest angle, and has a measure of $\boxed{30^\circ}$.

- (12) **-6** ID: [1C544]

Each instance where the graph of $f(x)$ crosses the x -axis gives us a value of x for which $f(x) = 0$. From the graph, we can see that this occurs at $x = -7$, $x = -2$, and $x = 3$. The sum of these x -values is $-7 - 2 + 3 = \boxed{-6}$.

- (13) **289 people** ID: [14AD3]

There are 12 different months. The most people you could have such that no 25 people have the same birth month is if 24 people shared each birth month. This gives a total of $12 \cdot 24 = 12 \cdot 12 \cdot 2 = 144 \cdot 2 = 288$. If we add one more person, no matter what their birth month is, there will have to be 25 with the same birth month. Thus, $288 + 1 = \boxed{289}$ is the minimum.

(14) 44 ID: [A0403]

If we let x and y be the two real numbers, we know that $\frac{x+y}{2} = 7 \Rightarrow x+y = 14$. We also know that the average of their squares is 54, so $\frac{x^2+y^2}{2} = 54 \Rightarrow x^2+y^2 = 108$. Now, instead of solving for x and y individually, which could involve some messy algebra, we try to find another way to find xy . We know that $(x+y)^2 = x^2 + 2xy + y^2$. We can now substitute in 14 for $(x+y)$ and 108 for (x^2+y^2) and solve for xy .

$$\begin{aligned}(x+y)^2 &= (x^2+y^2) + 2xy \\ \Rightarrow 14^2 &= 108 + 2xy \\ \Rightarrow 2xy &= 88 \\ \Rightarrow xy &= \boxed{44}\end{aligned}$$

(15) 4 cents ID: [1C]

Bill's bottle of soda contains 32 oz and costs 3 cents per ounce. Therefore, in total, Bill's bottle costs 32 oz $\cdot$ 3 cents = 96 cents.

Jill's bottle of soda contains 40 oz and costs 2.5 cents per ounce. Therefore, in total, Jill's bottle costs 40 oz $\cdot$ 2.5 cents = 100 cents.

Thus, Jill's bottle of soda costs $100 - 96 = \boxed{4 \text{ cents}}$ more than Bill's bottle of soda.

(16) 24/13 hours ID: [D8B54]

Let a, b, c represent how many jobs Alana, Bob, and Cody can respectively complete in one hour. From the information given, we can make three equations:

$$\begin{aligned}2(a+b) &= 1, \\ 3(b+c) &= 1, \\ 4(a+c) &= 1.\end{aligned}$$

Hence,

$$\begin{aligned}a+b &= \frac{1}{2}, \\ b+c &= \frac{1}{3}, \\ a+c &= \frac{1}{4}.\end{aligned}$$

We can solve for a, b , and c individually. But we are interested in what happens when Alana, Bob, and Cody work together, and their collective rate is $a+b+c$. To find this, we can add all three equations above, which gives us

$$2a + 2b + 2c = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

Hence,

$$a+b+c = \frac{13}{24}.$$

In other words, when Alana, Bob, and Cody all work together, they can complete $\frac{13}{24}$ of the job in one hour.

Therefore, completing the job takes them $\boxed{\frac{24}{13}}$ hours.

(17) 15 units ID: [01003]

It is well-known that if two objects A and B are similar by a similarity factor of r (i.e., if $r = 2$, then B is twice as big as A), then the area of B is r^2 times the area of A . Thus, if the larger right triangle has area $\frac{1}{2} \cdot 9 = 9$ times the area of the smaller triangle, it must be $\sqrt{9} = 3$ times as large. Thus the length of the hypotenuse of the larger triangle is $(3)(5) = \boxed{15}$ units.

(18) 961 ID: [CA0D4]

For a number n to have exactly three positive factors, it must be the square of a prime. This is because for $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, it will have $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ factors. Three is prime, so there can only be a single prime factor in the factorization, implying that n is of the form p^2 .

Since $30^2 = 900$, we know we should be looking for a prime number that's a bit larger than 30. $31^2 = 961$, and 37^2 has more than three digits. Thus, $\boxed{961}$ is our answer.

(19) 28% ID: [C3SD1]

For each possible hundreds digit (1, 2, 3, ..., 9), there are 100 possible three digit numbers that begin with that hundreds digit, for a total of $9 \times 100 = 900$ different three digit numbers. We now find how many of these three digit numbers contain the digit 5 at least once. All 100 of the three digit numbers with 5 in their hundreds place must have at least one 5. For each of the other eight possible hundreds digits, there are 10 three digit numbers with five in the tens place and 9 with five in the ones place but not the tens. Therefore, there are $8 \times 19 = 152$ three digit numbers with at least one five in their tens or ones places.

Adding this total to the number of three digit numbers with seven in their hundreds place, we have a total of $100 + 152 = 252$ such numbers. Therefore, the percentage of three digit numbers that contain at least one 5 is $\frac{252}{900} = \frac{7}{25} = \boxed{28\%}$.

(20) 37 ID: [DA424]

If we wish to maximize the second largest of these integers, then there must be one integer larger than it, and we must minimize the five smallest integers.

We can let the five smallest integers be 1, 2, 3, 4, 5 since they must be distinct and positive. They have a sum of 15 so far, so the other two integers must add up to $91 - 15 = 76$. We need to maximize the smaller of these two integers (the second largest of all seven), so we want these integers to be as close as possible. If they are far apart, then the largest number will be significantly larger, but the second largest number will be significantly smaller. With this in mind, the integers we seek are 37 and 39.

The second largest integer is maximized in the set $\{1, 2, 3, 4, 5, 37, 39\}$, and this second largest integer is $\boxed{37}$.