

Sprint 1

$$2 + (2 \times 2) + (2 \times 2 \times 2) = 2 + 4 + 8 = \mathbf{14}.$$

Sprint 2

There are 3 vowels (A, O, U), each worth 8, and 7 consonants (M, T, H, C, N, T, S), each worth 5. As a cross-check (it being easy to overlook a letter), let's count all the letters in MATHCOUNTS and verify it is $3 + 7 = 10$ —yes, keep going. The total value is, therefore, $3 \times 8 + 7 \times 5 = 24 + 35 = \mathbf{59}$.

Sprint 3

$$12x = 36, \text{ so } x = 36/12 = \mathbf{3}.$$

Sprint 4

Of the five values (21, 15, 12, 37, 48), only two (12, 48) are multiples of 6, so the probability is $\mathbf{2/5}$.

Sprint 5

Let A , B , and E be the current age of Al, Ben, and Ed, respectively. $E - B = (A + 2) - (A - 5) = (A - A) + (2 + 5) = \mathbf{7}$.

Sprint 6

$$\frac{12 - \heartsuit}{3} = 1. \text{ Multiply both sides by 3: } 12 - \heartsuit = 3. \text{ Add } \heartsuit - 3 \text{ to both sides: } \mathbf{9 = \heartsuit}.$$

Sprint 7

$$A = (2 \times 10 \text{ cm})(2 \times 15 \text{ cm}) = 20 \times 30 \text{ cm}^2 = \mathbf{600 \text{ cm}^2}.$$

Sprint 8

Because only positive values are involved, the least product is the product of the two least elements, 1 and 2, and the greatest product is the product of the two greatest elements, 3 and 4. Therefore, the desired sum is $1 \times 2 + 3 \times 4 = \mathbf{14}$.

Sprint 9

The figure suggests a linear relationship. When x shifts by 6 from 0 to 6, y shifts 4 from -4 to 0. Therefore, when x shifts another 6 to 12, y shifts another 4 from 0 to $\mathbf{4}$.

Sprint 10

A sphere inscribed in a cube is tangent to each face of the cube. Any line segment connecting the points of tangency on opposite faces of the cube constitutes a diameter of the sphere. Therefore, both the diameter of the sphere and edge-length of the cube are the same, 6 cm. The radius of the circle is half that diameter, thus 3 cm. Therefore, the volume enclosed by the sphere is given by $V_{\text{sphere}} = \frac{4}{3}\pi(3 \text{ cm})^3 = \frac{4\pi \times 27}{3} \text{ cm}^3 = \mathbf{36\pi \text{ cm}^3}$.

Sprint 11

$$\frac{15+7+12+5+x}{5} = 10, \text{ so } x = 5 \times 10 - 15 - 7 - 12 - 5 = \mathbf{11}.$$

Sprint 12

The volume of a rectangular prism is the square root of the product of the surface areas of three mutually adjacent faces (all three faces intersecting at one vertex). For any such set of three faces, one face has area 35 in^2 , another has area 45 in^2 , and yet another has area 63 in^2 . The numeric parts have respective prime factorizations of 5×7 , $3^2 \times 5$, $3^2 \times 7$. Therefore, we need

$$\sqrt{3^4 \times 5^2 \times 7^2 (\text{in}^2)^3} = 3^2 \times 5 \times 7 \text{ in}^3 = 9 \times 5 \times 7 \text{ in}^3 = 45 \times 7 \text{ in}^3 = \mathbf{315 \text{ in}^3}.$$

Sprint 13

The only way to use quarters in 1 quarter is with 1 nickel. Without quarters, there must be 0, 1, 2, or 3 dimes, with the only choice for nickels in each case being 6 minus twice the number of dimes. The total number of coin distributions is, therefore, **5**.

Sprint 14

Use the complement technique—shaded area is total area minus unshaded areas. The total area is $4\text{ cm} \times 5\text{ cm} = 20\text{ cm}^2$. The upper left unshaded area is $3\text{ cm} \times 3\text{ cm}/2 = 4.5\text{ cm}^2$. The lower left unshaded area is $1\text{ cm} \times 5\text{ cm}/2 = 2.5\text{ cm}^2$. The upper right unshaded area is $4\text{ cm} \times 2\text{ cm}/2 = 4\text{ cm}^2$. Therefore, the shaded area is $(20 - 4.5 - 2.5 - 4)\text{ cm}^2 = \mathbf{9\text{ cm}^2}$.

Sprint 15

The total count of marbles is $12 + 7 + 11 + 3 = 33$. The first selection is to be pink, which is 3 desired choices out of 33 total choices, so $\frac{3}{33} = \frac{1}{11}$. Now there are 32 marbles left to choose from, of which 12 are the desired purple, so $\frac{12}{32} = \frac{3}{8}$. These two events are independent, so the overall probability is the product of these two individual probabilities: $\frac{1}{11} \times \frac{3}{8} = \frac{3}{88}$.

Sprint 16

The tank volume is $10\text{ ft} \times 3\text{ ft} \times 2\text{ ft} = 60\text{ ft}^3$. At 11:00 the water volume is $70\% = 0.7$ of the tank volume, thus $0.7 \times 60\text{ ft}^3 = 42\text{ ft}^3$. After 1 h, it is down to $0.4 \times 60\text{ ft}^3 = 24\text{ ft}^3$, thus dropping $42\text{ ft}^3 - 24\text{ ft}^3 = 18\text{ ft}^3$ each hour. Therefore, after 1 h more, the water volume will drop another 18 ft^3 to $24\text{ ft}^3 - 18\text{ ft}^3 = \mathbf{6\text{ ft}^3}$.

Sprint 17

$2025 = 81 \times 25 = 3^4 \times 5^2$, so the requisite sum is $3 + 4 + 5 + 2 = \mathbf{14}$.

Sprint 18

Each path starts at 5. Then there are 4 choices: squares 2, 4, 6, or 8. For each of those, there are 2 adjacent corner squares that can be chosen. Then, no matter how you got to where you are, there is only 1 unused adjacent square. This yields a total of $4 \times 2 \times 1 = 8$ distinct paths. However, these 8 paths are paired based on covering the same four squares, thus having the same sum, and merely swapping which square is covered by move 1 versus move 3. This results in 4 distinct patterns for sum: the 2×2 squares in (1) the upper-left corner, (2) the upper-right corner, (3) the lower-left corner, (4) the lower-right corner. The sum of the respective sums is:

$$(1 + 2 + 4 + 5) + (2 + 3 + 5 + 6) + (4 + 5 + 7 + 8) + (5 + 6 + 8 + 9) = 12 + 16 + 24 + 28 = \mathbf{80}.$$

Sprint 19

The amount of increase in the average is the average increase in the ten values is

$$\frac{(1+2+3+4+5+6+7+8+9+10)}{10} = \frac{10 \times 11/2}{10} = 5.5, \text{ so the new average is } 9.5 + 5.5 = \mathbf{15}.$$

Sprint 20

The prime factorization of 64 is 2^6 ; that exponent 6 has 4 factors: 6, 3, 2, 1, so we have $(2^6)^1 = 64^1$; $(2^3)^2 = 8^2$; $(2^2)^3 = 4^3$; $(2^1)^6 = 2^6$. Desired sum is $(64 + 1) + (8 + 2) + (4 + 3) + (2 + 6) = \mathbf{90}$.

Sprint 21

The first three positive integers violate the triangle inequality theorem, thus do not form a triangle. With 2 and 3 available, the next length needs to be at least $2 + 3 = 5$ to violate the triangle inequality theorem, so pick 5 for bare minimum violation. Similarly, with 3 and 5 available, the last length needs to be at least $3 + 5 = 8$. We are generating the Fibonacci sequence starting with 1 and 2. The five needed values are 1; 2; 3; 5; 8. Two ways to add these: (1) There are only five values and they are small integers, so just add them. (2) The sum of a general contiguous finite subsequence of Fibonacci values is given by $F_m + F_{m+1} + \cdots + F_{n-1} + F_n = F_{n+2} - F_{m+1}$ —if you have the Fibonacci sequence memorized to at least 21 or the problem involved more than five matches, this would be faster, but for this problem, we have $5 + 8 = 13$; $8 + 13 = 21$, and the second one in our list is 2, so subtract $21 - 2 = 19$. Therefore, with all lengths being in centimeters, the desired sum is **19** cm.

Sprint 22

For $2^x = x + 3$, $2^x > 0$ for all real numbers x , so the equation requires $x + 3 > 0$. Dividing both sides of the equation by the nonzero value $x + 3$ yields $\frac{2^x}{x + 3} = 1$, so

$$\frac{2^{2^x}}{x + 3} = \frac{2^{(2^x)}}{x + 3} = \frac{2^{(x+3)}}{x + 3} = \frac{2^x \cdot 2^3}{x + 3} = \frac{2^x}{x + 3} \cdot 8 = 1 \times 8 = \mathbf{8}.$$

Sprint 23

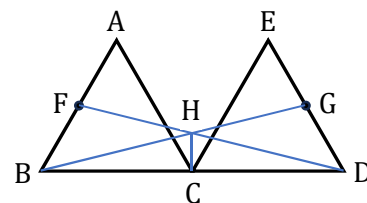
Square both sides of $\sqrt{6 + 2\sqrt{5}} = a + \sqrt{b}$ yields $6 + 2\sqrt{5} = a^2 + b + 2a\sqrt{b}$, so $b = 5$ and $a = 1$. (With these values for a and b , $a + \sqrt{b} > 0$ and $\sqrt{}$ applied to a positive value always yields a positive value, so the signs of the two sides match and squaring both sides will not trigger any extraneous “solutions”.) Therefore, $a + b = 1 + 5 = \mathbf{6}$.

Sprint 24

Because $n > 0$, $\frac{2}{3} \leq \frac{m}{n} < 1$ is equivalent to $\frac{2}{3}n \leq m < n$. Because m and n are integers, we can tighten the last pair of inequalities to $\left\lceil \frac{2}{3}n \right\rceil \leq m \leq n - 1$ (where $\lceil \square \rceil$ denotes the ceiling operator, the least integer greater than or equal to the one operand, which is enclosed between the two parts of the operator). Thus, the number of satisfactory values of m for any particular value of n is given by $n - \left\lceil \frac{2}{3}n \right\rceil$, which is positive only for $n \geq 3$. For n of the form of any of $3k, 3k + 1, 3k + 2$ for some integer k , there are k values of m that satisfy the constraints. For $3 \leq n \leq 10$, n ranges from $3(1)$ through $3(3) + 1$, yielding 3 with $k = 1$, 3 with $k = 2$, and 2 with $k = 3$. Therefore, there are in total $3 \times 1 + 3 \times 2 + 2 \times 3 = \mathbf{15}$ distinct (n, m) pairs satisfying the given constraints.

Sprint 25

Establish a coordinate system with the x -axis along $\overline{BD}$ and origin at C. All coordinates represent number of inches for x and y . Then A is at $(-7, 7\sqrt{3})$; B is at $(-14, 0)$; C is at $(0, 0)$; D is at $(+14, 0)$; E is at $(+7, 7\sqrt{3})$; F is at $(-10.5, 3.5\sqrt{3})$; G is at $(+10.5, 3.5\sqrt{3})$. The slope of $\overline{BG}$ is $\frac{3.5\sqrt{3}-0}{10.5-(-14)} = \frac{\sqrt{3}}{7}$. The horizontal distance traversed from B to



H is $[0 - (-14)]$ in = 14 in; multiplying that by the slope $\frac{\sqrt{3}}{7}$ yields a y -coordinate at H of $0 + 2\sqrt{3} = 2\sqrt{3}$. Therefore, H is at $(0, 2\sqrt{3})$, which is a distance of $2\sqrt{3}$ in above C.

Sprint 26

Let's define $a' = \frac{a-1}{2}$, $b' = \frac{b-1}{2}$, $c' = \frac{c-1}{2}$, so that we are dealing with consecutive nonnegative integers so as not having to compensate for skips, etc. By now requiring $a' + b' + c' = \frac{49-3}{2} = 23$ with $0 \leq a' < b' < c'$, we have a parallel case with the same total counts. Now, $0 \leq a' \leq \left\lfloor \frac{23-3}{3} \right\rfloor = 6$. For $a' = 6$, (b', c') ranges from $(7, 10)$ to $(8, 9)$ [2 cases]. For $a' = 5$, (b', c') ranges from $(6, 12)$ to $(8, 10)$ [3 cases]. For $a' = 4$, (b', c') ranges from $(5, 14)$ to $(9, 10)$ [5 cases]. For $a' = 3$, (b', c') ranges from $(4, 16)$ to $(9, 11)$ [6 cases]. For $a' = 2$, (b', c') ranges from $(3, 18)$ to $(10, 11)$ [8 cases]. For $a' = 1$, (b', c') ranges from $(2, 20)$ to $(10, 12)$ [9 cases]. For $a' = 0$, (b', c') ranges from $(1, 22)$ to $(11, 12)$ [11 cases]. The total is $2 + 3 + 5 + 6 + 8 + 9 + 11 = 44$ valid ordered triples.

Sprint 27

The k th term of the series is $4!_{k+3}C_4$. The sum of the first m such terms is $4!_{m+4}C_5$. Thus, for $m = 25$, the sum is $4! \times \frac{29 \times 28 \times 27 \times 26 \times 25}{5!} = \frac{29 \times 28 \times 27 \times 26 \times 25}{5} = 29 \times 28 \times 27 \times 26 \times 5$. The 28 provides 2 factors of 2 and the 26 provides 1 factor of 2, for a total of 3 factors of 2. Therefore, $n = 3$.

Sprint 28

$76^2 - 75^2 = (76 + 75)(76 - 75) = 151 \times 1 = 151$. Similarly, $75^2 - 74^2 = 75 + 74 = 149$. Therefore, by linear interpolation, for $74^2 \leq n \leq 75^2$, $\sqrt{n+1}$ averages about $1/149$ greater than $\sqrt{n}$; for $75^2 \leq n \leq 76^2$, $\sqrt{n+1}$ averages about $1/151$ greater than $\sqrt{n}$. We have n right at 75^2 , the boundary between these two intervals, so it is reasonable to expect that $\sqrt{75^2 + 1}$ is very close to $1/150$ greater than $\sqrt{75^2} = 75$. Now, $\frac{1}{150} = \frac{1}{25} \times \frac{1}{6} = \frac{0.04}{6} = 0.00\overline{6}$, so $\sqrt{75^2 + 1} \approx 75.00\overline{67}$ (true value is $75.006\ 666\ 3\dots$). The first nonzero digit to the right of the decimal mark is 6.

Sprint 29

This technique is sometimes referred to as Simon's [Rubinstein-Salzedo] Favorite Factoring Trick. Let's adjust the satisfying equation by multiplying both sides by $(a + b)$ to get $7ab = 2025(a + b)$. Now expand the right side and shift both terms to the left side: $7ab - 2025a - 2025b = 0$. The equation is symmetric in a and b . We need to do some factoring somewhat similar to completing the square, but 7 is prime. To maintain the symmetry for factoring, let's multiply both sides of the equation by 7: $0 = 7^2ab - 7(2025a) - 7(2025b) = 7a(7b - 2025) - 2025(7b - 2025) - 2025^2 = (7a - 2025)(7b - 2025) - 2025^2$. Add 2025^2 to both sides of the equation: $(7a - 2025)(7b - 2025) = 2025^2 = 3^8 \times 5^4$. Now we need to decompose $3^8 \times 5^4$ into all possible pairs of factors whose product is $3^8 \times 5^4$ such that one factor corresponds to $(7a - 2025)$ and the other factor to $(7b - 2025)$. There are $(8 + 1)(4 + 1) = 45$ positive divisors of $3^8 \times 5^4$. One of them is 45, which pairs with itself. The remaining 44 divisors form 22 pairs if we ignore order of factors. If one factor in a pair matches up, then so does the other factor. Thus, we need consider only the 23 divisors less than or equal to $3^4 \times 5^2 = 2025$, and determine how many of them match, then double that count because for $a < b$ both (a, b) and (b, a) qualify as distinct solutions for this problem, except that if $(45, 45)$ works, then it is a special case yielding only one solution, not two. Each factor must match with one of $(7a - 2025)$ or $(7b - 2025)$, each satisfying $(7x - 2025) \equiv 5 \pmod{7}$. Therefore, we will count those numeric divisors of $3^8 \times 5^4$ whose remainder upon dividing by 7 is 5. The divisors of $3^8 \times 5^4$ less than or equal to 2025 are, with those satisfying the modular congruence in italic: 1; 3; 9; 27; 81; 243; 729; 5; 15; 45; 135; 405; 1215; 25; 75; 225; 675; 2025; 125; 375; 1125; 625; 1875. There are 4 qualifying divisors, thus 8 suitable ordered pairs.

Sprint 30

Regard $\triangle ABC$ as the base of the tetrahedron and BD as its height, given that $\overline{BD}$ is perpendicular to the base. A tetrahedron is a pyramid composed of a triangular base and three lateral triangular sides, and its volume is $1/3$ of the product of the area of the base times the height. Because $m\angle ABC = 90^\circ$, the base is a right triangle and its area is $BA \cdot BC/2$. The given angular measures indicate that $\triangle ABD$ and $\triangle BCD$ are 30° - 60° - 90° triangles, making the longer leg $\sqrt{3}$ times as long the shorter leg. $m\angle DAB = 30^\circ$ means BD is the shorter leg and BA is the longer leg, so $BA = \sqrt{3}BD$. $m\angle DCB = 60^\circ$ means BC is the shorter leg and BD is the longer leg, so $BC = BD/\sqrt{3}$. Therefore, the volume of the tetrahedron is $\frac{1}{3} \left(\frac{1}{2} BA \cdot BC \right) \cdot BD = \frac{1}{6} \left[(\sqrt{3} BD) \cdot \left(\frac{BD}{\sqrt{3}} \right) \right] \cdot BD = \frac{1}{6} (BD)^3 = 2025 \text{ cm}^3$. Therefore, $BD = \sqrt[3]{6 \times 2025 \text{ cm}^3} = \sqrt[3]{2 \times 3^5 \times 5^2 \text{ cm}^3} = 3\sqrt[3]{2 \times 3^2 \times 5^2} \text{ cm} = 3\sqrt[3]{450} \text{ cm}$.