2018 Chapter Competition Sprint Round

1. The next integer down from 2017.7012 is 2017, which is 0.7012 below; the next integer up is 2018, which is 0.2988 above. The latter is the lesser difference, so **2018** is the closest.

2. The minute hand (the longer hand) is pointing directly at the 7. Each increment of a number on the face corresponds to 5 minutes for the minute hand. Therefore, the clock is 7×5 minutes = **35** minutes after the hour.

3. Start with \$835;
   subtract \$415 for rent,
   leaving \$420;
   subtract \$220 for utilities,
   leaving \$**200**.

4. The halfway point occurs at the average of the two endpoints, which are at 0 and 8, so $p = (0 + 8)/2 = \mathbf{4}$.

5. $2 \times \$14$ and $14 \times \$2$ both equal $\$2 \times 14 = \$28$. Since we have this cost twice, the total is $2 \times \$28 = \$\mathbf{56}$.

6. $n + 4 = 10 - 2 = 8$. Therefore, $n = 8 - 4 = \mathbf{4}$.

7. a for apples and p for pears: $a/p = 3/5 = a/20$, so $a = 20 \times 3/5 = 4 \times 3 = \mathbf{12}$ apples.

8. $462 = 7 \times 11 \times p$, so $p = (462/11)/7 = 42/7 = \mathbf{6}$.

9. $1/0.25 - 2 = 1/(1/4) - 2 = 4 - 2 = \mathbf{2}$.

10. There are 5 values, so the arithmetic mean is the sum of all the values then divided by 5:
    $10 = (9 + 11 + 13 + x + 7)/5 = (40 + x)/5$, so $40 + x = 5 \times 10 = 50$.
    Therefore, $x = 50 - 40 = \mathbf{10}$.

11. $3(p + 4) + 4(p + 11) = 7(p + q)$;
    $3p + 12 + 4p + 44 = 7p + 7q$;
    $7p + 56 = 7p + 7q$;
    $56 = 7q$;
    $q = \mathbf{8}$.

12. $V = hwd = 10\text{ inches} \times 8\text{ inches} \times 3\text{ inches} = (10 \times 8 \times 3) = \mathbf{240}\text{ in}^3$.

13. $\frac{z}{16} = \frac{4}{z}$. Multiply both sides by $16z$:
    $z^2 = 64$, so $z = \pm 8$.
    We are directed to take the positive value, so $z = \mathbf{8}$.

14. She has $8 - 1 = 7$ non-rotten apples costing \$3.50.
Therefore, each non-rotten apple cost $\$3.50/7 = \$0.50 =$ **50** cents.

15. $123{,}456 + 1980 = 125{,}436$, which has swapped the 3 and the 5.
$3 \times 5 =$ **15**.

16. The sum of the measures of the three angles of a triangle is always 180 degrees. Two of the angles are 35 degrees and 95 degrees, totaling 130 degrees, so the remaining angle must have measure 180 degrees – 130 degrees = **50** degrees.

17. The values of the 25¢, 10¢, 5¢ and 1¢ are separated enough that duplicated sums cannot occur. In any possible mix of the coins, the quarter can be included or excluded (2 options), likewise for the dime (2 options), the nickel (2 options), and the penny (2 options). Each of these 4 inclusion-exclusion cases is independent of the others, so there are $2 \times 2 \times 2 \times 2 = 16$ possible mixes of coins; however, one of these, excluding all 4 coins, does not meet the criterion that at least one coin must be used. Therefore, there are **15** possible amounts.

18. $x + y = 14$ and $x - y = 4$.
The sum of these two equations is $2x = 18$, so $x = 9$.
Thus, $9 + y = 14$, so $y = 5$.
Therefore, $xy = 9 \times 5 = \mathbf{45}$.

19. $x^2 - y^2 = (x + y)(x - y)$. Let $x = 16$ and $y = 15$.
Therefore, $16^2 - 15^2 = (16 + 15)(16 - 15) = 31 \times 1 = \mathbf{31}$.

20. The mean of x and y is $\frac{x+y}{2}$. Therefore,
$\frac{a+b}{2} = 6.2$;
$\frac{b+c}{2} = 7.3$;
$\frac{a+c}{2} = 4.5$.
We see that each of $a/2$, $b/2$ and $c/2$ occurs twice in these three equations, so if we add the three equations together, we will get the desired $a + b + c$.
Therefore, $a + b + c = 6.2 + 7.3 + 4.5 = \mathbf{18}$.

21. If you know the first few powers of 3, this question is very easy and fast. For those who do not know that many powers of 3, we have $3^2 = 9$; $9^2 = 81$ and $9^2 = (3^2)^2 = 3^4$, so $3^4 = 81$. We are still below 100 but not by much. The next power of 3 is 243, which is too big.
Therefore, the largest integer exponent is **4**.

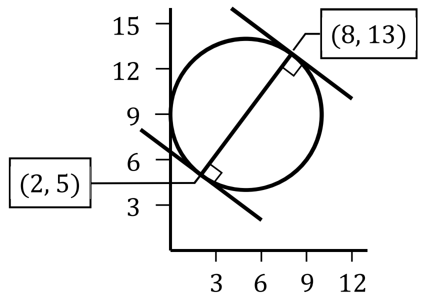
22. A number is divisible by 6 if and only if it is divisible by 2 and by 3. To be divisible by 2, the units digit must be even, so A must be 0, 2, 4, 6 or 8. To be divisible by 3, the sum of all the digits in the number must be divisible by 3, so $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + \text{A} = 44 + \text{A}$ needs to be 45, 48 or 51 to be divisible by 3 and have A be a digit. That means A must be 1, 4 or 7. The only value that satisfies both divisibility requirements is **4**.

23. We can make rectangles 1 column wide, in which case we have 3 choices, 2 columns wide, in which case we have 2 choices (#1 with #2 or #2 with #3), or all 3 columns wide, with only 1 choice. Thus, we have $3 + 2 + 1 = 6$ choices for arranging columns to make rectangles. The 3 rows work the same way as the 3 columns, so there are 6 choices for arranging rows. The manipulation of rows is independent of the manipulation of columns (for each of the 6 row choices, we have 6 column choices), making a total of $6 \times 6 = \mathbf{36}$ rectangles.
More generally, if you have r rows, there will be T_r choices of row arrangements, where T_r is the rth triangular number, $r(r + 1)/2$. Likewise, with c columns, there will be T_c choices of column arrangements, for a total of T_rT_c rectangles. In this problem, $T_3T_3 = 6 \times 6$.

24. The denominator of each of the 10 factors in $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{15}{16}\cdot\frac{17}{18}\cdot\frac{19}{20}$ is even and, therefore, has a factor of 2. Let's pull out the 10 factors of 2 in the denominators to yield: $\frac{1}{2^{10}}\cdot\frac{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15\cdot17\cdot19}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}$, with both the numerator and denominator having factors 1, 3, 5, 7, and 9 that can be cancelled out, leaving $\frac{1}{2^{10}}\cdot\frac{11\cdot13\cdot15\cdot17\cdot19}{2\cdot4\cdot6\cdot8\cdot10}$. Again each of the 5 factors in the denominator of the fraction on the right is even and has a factor of 2 that can be pulled out, so let's remove them from the right fraction and give them to the denominator or the left fraction, now making 15 factors of 2 there: $\frac{1}{2^{15}}\cdot\frac{11\cdot13\cdot15\cdot17\cdot19}{1\cdot2\cdot3\cdot4\cdot5}$. Now, the factors of 3 and 5 together in the denominator on the right cancel with the 15 in the numerator, leaving $\frac{1}{2^{15}}\cdot\frac{11\cdot13\cdot17\cdot19}{1\cdot2\cdot4}$. The denominator on the right is now $8 = 2^3$, so we have 3 more factors of 2 to move to the 15 already in the denominator on the left, yielding $\frac{1}{2^{18}}\cdot 11\cdot 13\cdot 17\cdot 19 = \frac{11\cdot13\cdot17\cdot19}{2^{18}}$. Now we have some arithmetic to do and this is Sprint Round—only the biological calculator above your shoulders, with assistance of pencil and paper, allowed. There are several ways to do this efficiently, and different people may well have different preferences. One way is to rearrange:
$11\cdot 13\cdot 17\cdot 19 = (13\cdot 17)(11\cdot 19) = [(15 + 2)(15 - 2)][(15 + 4)(15 - 4)]$
$= (15^2 - 2^2)(15^2 - 4^2) = (15^2)^2 - (4 + 16)15^2 + (4 \times 16)$
$= 15^2(15^2 - 20) + 64 = 225 \times 205 + 64 = (215 + 10)(215 - 10) + 64$
$= 215^2 - 10^2 + 64.$
Now we need to square an integer 215 ending in 5. Break off that 5 and call the rest of the number n, so here $n = 21$. The square of such a number can be calculated is the number of 100s being $n(n + 1)$ and the attaching 25 for the tens and units digits:
$21(21 + 1) = 21 \times 22 = 21 \times 2 \times 11 = 42 \times 11 = 462$, using the multiply by 11 trick.
Therefore, $215^2 = 46\,225$, so we need $46{,}225 - 100 + 64 = 46{,}125 + 64 = 46{,}189$, to which we need to add the exponent 18 that 2 was raised to: $46{,}189 + 18 = \mathbf{46,207}$.

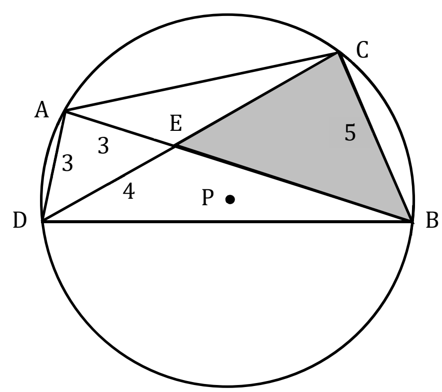
25. Let t be the total bonus amount; A be Arman's portion, B be Bernardo's portion, and C be Carson's portion.
$A = \frac{1}{3}t + \$10.$
$B = \frac{1}{2}(t - A) + \$3 = \frac{1}{2}\left[t - \left(\frac{1}{3}t + \$10\right)\right] + \$3 = \frac{1}{3}t - \$2.$
$C = \$25.$
$t = A + B + C = \frac{1}{3}t + \$10 + \frac{1}{3}t - \$2 + \$25 = \frac{2}{3}t + \$33,$
so $\frac{1}{3}t = \$33$ and $t = \$\mathbf{99}$.

26. A tangent line to a circle is always perpendicular to the radius connecting the center of the circle to the point of tangency. We have two distinct tangent lines with the same slope, meaning the circle is between the two tangent lines. Each corresponding radius must be perpendicular to its tangent line and have slope equal to the negative reciprocal of the two tangent lines. The two radii are distinct, have the same slope, and coincide at the center of the circle, which means the two radii combined form a diameter of the circle. We know the two endpoints of the diameter, so we can determine its slope: $(13 - 5)/(8 - 2) = 8/6 = 4/3$. The two tangents lines are perpendicular to this diameter and have slope equal to the negative reciprocal of the diameter's slope, thus $\mathbf{-3/4}$.



27. Let the four values be $10 \le a \le b \le c$. The mean is $\frac{10+a+b+c}{4}$. The median is $\frac{a+b}{2}$. The range is $c - 10$. The mean being equal to the median implies $\frac{10+c}{4} = \frac{a+b}{4}$, so $10 + c = a + b$. The median being equal to the range implies $a + b = 2(c - 10) = 2c - 20$. Therefore, $10 + c = 2c - 20$, so $c = \mathbf{30}$.

28. Most MATHCOUNTS students are familiar with the intersecting chords theorem. The basis of the proof for that theorem is that triangles ADE and CBE are similar. The scale factor of similarity in this case is that segment BC = 5 and corresponding segment AD = 3, so the linear dimensions of CBE are 5/3 times the corresponding linear dimensions of ADE, and the area of CBE is $(5/3)^2 = 25/9$ times the area of ADE. We can find the area of ADE by Heron's formula since we know the lengths of all three sides: 3, 3 and 4. The semi-perimeter is $(3 + 3 + 4)/2 = 5$, so the area of ADE is given by: $\sqrt{5(5-3)(5-3)(5-4)} = \sqrt{20} = 2\sqrt{5}$. The area CBE is 25/9 times this, or $\mathbf{\frac{50\sqrt{5}}{9}}$ units².



29. $\frac{1}{6} = \left|\frac{x-2018}{x-2019}\right| = \left|1 + \frac{1}{x-2019}\right|$.

The absolute value introduces two cases:

Case 1: $1 + \frac{1}{x-2019} = \frac{1}{6}$, so $\frac{1}{x-2019} = \frac{1}{6} - 1 = -\frac{5}{6}$ and $x - 2019 = -\frac{6}{5}$.

Therefore, $x = 2019 - \frac{6}{5}$. (I am leaving it in this form for reasons to be seen shortly.)

Case 2: $1 + \frac{1}{x-2019} = -\frac{1}{6}$, so $\frac{1}{x-2019} = -\frac{1}{6} - 1 = -\frac{7}{6}$ and $x - 2019 = -\frac{6}{7}$.

Therefore, $x = 2019 - \frac{6}{7}$.

Combining cases: We want $\left|\left(2019 - \frac{6}{5}\right) - \left(2019 - \frac{6}{7}\right)\right| = \left|\frac{6}{7} - \frac{6}{5}\right|$ since the 2019's cancel each other.

$\left|\frac{6}{7} - \frac{6}{5}\right| = \left|\frac{6\times5-6\times7}{7\times5}\right| = \left|-\frac{12}{35}\right| = \boldsymbol{\frac{12}{35}}$.

30. Each die can independently show any integer from 1 to 6, so the sum S can range from 6 (all 1s) to 36 (all 6s).

For $S = 36$, $S(42 - S) = 36 \times 6 = 216$ uses all 6s with probability $\frac{6!}{6!}\left(\frac{1}{6}\right)^6 = 1\left(\frac{1}{6}\right)^6$.

For $S = 35$, $S(42 - S) = 35 \times 7 = 245$ uses five 6s and one 5 with probability $\frac{6!}{5!1!}\left(\frac{1}{6}\right)^6 = 6\left(\frac{1}{6}\right)^6$.

For $S = 34$, $S(42 - S) = 34 \times 8 = 272$ uses *either* five 6s and one 4 with probability $6\left(\frac{1}{6}\right)^6$ or four 6s and two 5s with probability $\frac{6!}{4!2!}\left(\frac{1}{6}\right)^6 = 15\left(\frac{1}{6}\right)^6$.

For $S = 33$, $S(42 - S) = 33 \times 9 = 297$ does not qualify.

Remember, commutativity of multiplication applies and the reverse cases apply, such as 6×36 for $S = 6$. The same probability applies for $S = 6, 7$ and 8 as for $S = 36, 35$ and 34, respectively, so we need to double the sum of the first three case sets. Therefore, the total probability is

$2(1 + 6 + 6 + 15)\left(\frac{1}{6}\right)^6 = 56\left(\frac{1}{6}\right)^6 = 7 \times 2^3\left(\frac{1}{6}\right)^3\left(\frac{1}{6}\right)^3 = 7\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^3 = \frac{7}{18^3}$

$= \boldsymbol{\frac{7}{5832}}$.