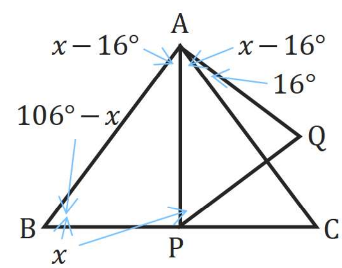
2020 State Team Round Solutions

1. The leftmost digit must be nonzero and even, and we want that digit as small as possible, which means 2. That leaves a needed sum of 12 for the remaining 4 digits, which has to be split across at least 2 digits. We must not forget that we may now use 0, and may use as many as two of them (remembering we need 2 more nonzero digits to contribute 12 to the sum), so let's make the next 2 digits 00. With the restriction of even and a sum of 12, the only options for the remaining digits are 48, 66 and 84, of which 48 is the smallest. Therefore, our desired number is **20,048**.

2. There are $2(20+12)-4=60$ posts (subtracting the 4 compensates for double counting the 4 corner posts, and, because the posts are placed in the form of a closed figure, there need to be 60 connections between posts for each wire. For each interpost and circumpost pair, $12+0.5=12.5$ of wire is needed. There are 60 such pairs for each wire and there are 3 wires. Therefore, the total length of wire needed is $3\times 60\times 12.5=\mathbf{2250}$ meters.

3. Let t represent the fraction of attempts at field goals worth 3 points; then $1-t$ represents the fraction of attempts at field goals worth 2 points, since all field goal are in one of those two categories. The average field goal point effectiveness is the sum over the two categories of the product of the point value of the category times the fraction of success for that category time the fraction of field goal attempts in that category: $1.0389=2\times 0.4993(1-t)+3\times 0.3651t=0.9986+0.0967t$, so $0.0967t=0.0403$ and $t=\frac{0.0403}{0.0967}=\frac{403}{967}=0.4167\ldots\times 100\ \%=41.67\ldots\ \%$, which rounds to **42**%.

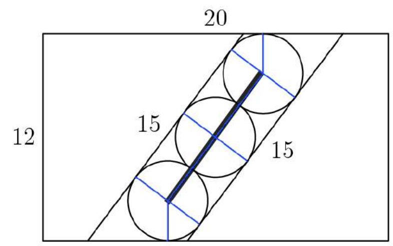
4. The isosceles triangles BAC and AQP are similar, so their base angles (PAQ, APQ, CBA and BCA) are congruent, and it the measure of the first, PAQ, that we are to determine. Let's call its measure x. Since we are given the measure of ∠CAQ as 16 degrees, the measure of ∠CAP is $x-16$. Now, since $\overline{\text{AP}}$ bisects the apex ∠CAB, that means that the measure of ∠CAB is $2x-32$ and each base angle has measure $90-\frac{1}{2}(2x-32)=106-x$; however, we also set the measure of the base angles to be x. Therefore, $x=106-x$, so $2x=106$ and $x=\mathbf{53}$ degrees.



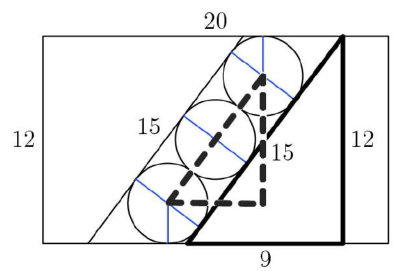
5. $(-8x;-8y)=☆(x,y)=(x+3y;9x-5y)$. For both components, the equality implies $9x+3y=0$, so $x=-\frac{1}{3}y$ and $\frac{x}{y}=-\mathbf{\frac{1}{3}}$.

6. When a point is rotated about a second point, the first point maintains a constant distance from the second point through the rotation from the pre-rotation position to the post-rotation. That means the pre-rotation position and the post-rotation position lie on a circle centered at the point about which the rotation is occurring. Therefore, the center point of the rotation lies on the perpendicular bisector of the line segment between the pre-rotation and post-rotation positions. A and A′ lie on the line $x=1$ and their midpoint is at $(1,6)$, so the perpendicular bisector is $y=6$. B and B′ lie on the line $y=8$ and their midpoint is at $(11,8)$, so the perpendicular bisector is $x=11$. The only point satisfying both equations is **(11, 6)**.

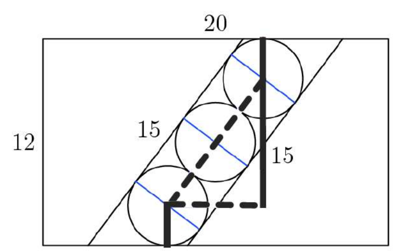
7. Let's call the radius of the circle r, and let's draw in all of the radii that might be useful:



Notice that the segment in the middle connecting the center of the bottom circle to the center of the top circle has length $4r$ and is parallel to the diagonal segments of length 15. And in particular, it is the hypotenuse of a right triangle that's similar to a 9-12-15 right triangle, as shown here:



This means that the dashed triangle has a height of $\frac{12}{15} \cdot 4r = \frac{16}{5}r$.



The total length of the solid vertical lines is $r + \frac{16}{5}r + r = \frac{26}{5}r$. But this must equal the height of the box, so $\frac{26}{5}r = 12$. Solving for r, we get $r = 12 \cdot \frac{5}{26} = \frac{60}{26} = \mathbf{\frac{30}{13}}$ feet. (Note that the length of the box was totally irrelevant. As long as the box is wide enough to fit the diagonal stripe, its length doesn't play any role in the solution.)

8. There are 2 ways for the addition of distinct primes to result in 10: 2 + 3 + 5 and 3 + 7.
For 2 + 3 + 5 we need numbers of the form $2^i 3^j 5^k$ where i, j and k are positive integers. The least such value is 30 and, indeed, all suitable numbers must be appropriate multiples of 30, that is of the form $30m$; m must consist only of factors of 2, 3 and 5 (not necessarily all, or even any, of those), and must be no more than 33 so that we do not exceed 1000. The acceptable values for m are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30 and 32—a count of 19.
For 3 + 7 we need numbers of the form $3^i 7^j$ where i and j are positive integers. The least such value is 21 and, indeed, all suitable numbers must be appropriate multiples of 21, that is of the form $21m$; m must consist only of factors of 3 and 7 (not necessarily both, or even either, of those), and must be no more than 47 so that we do not exceed 1000. The acceptable values for m are 1, 3, 7, 9, 21 and 27—a count of 6.
Therefore, the total count of integers from 2 to 1000, inclusive, with prime-sum radical being 10 is $19 + 6 = \mathbf{25}$ integers.

9. The tallest cheerleader must go in the middle of the back row. Randomly distribute the remaining 13 cheerleaders in the remaining 13 slots—that is 13! possible arrangements. Compare the leftmost cheerleaders in the two rows; there is a probability of $\frac{1}{2}$ that the taller is properly in the back row. Move one file to the right and again compare front and back row cheerleaders, with probability $\frac{1}{2}$ the taller of the two is in back. The same $\frac{1}{2}$ occurs for the third file. We must be careful now for the fourth (middle file), because we have already forced the tallest cheerleader to be in back at this location, so that file is guaranteed to have the taller in back. For each of the fifth, sixth, and rightmost files we have the probability $\frac{1}{2}$ of having the taller in the back row. Thus, we have a probability of $\left(\frac{1}{2}\right)^3 \times 1 \times \left(\frac{1}{2}\right)^3 = \frac{1}{64}$ of a taller cheerleader behind a shorter in each of the files. Now, there are $3! = 6$ arrangements of the leftmost 3 cheerleaders in the back row, with only 1 having the appropriate height drop-off going away from center, the probability of the correct arrangement for the back left is $\frac{1}{6}$. The same is true for the back right. Each of these considerations is independent of the others. Therefore, the number of distinct acceptable arrangements is $\frac{13!}{64\times 6\times 6} = \mathbf{2,702,700}$ ways.

10. Because we are dealing with remainders when products are divided by 30 and the count 900 of integers we have in scope is divisible by 30, we learn everything we need from modulo 30 arithmetic and processing only the integers 1 through 30; every other set of 30 consecutive integers behaves identically. Now, $30 = 2 \times 3 \times 5$ and of those prime factors, both 3 and 5 are relatively prime to the desired remainder value 4; if any of the 6 numbers multiplied together have 3 or 5 as a factor, then the product will have 3 or 5, respectively, as a factor, and that just does not happen with a remainder of 4 upon division by 30. Thus, we can throw away all the multiples of 3 or 5 from further consideration and analyze only 1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19, 22, 23, 26, 28 and 29 (a count of 16 values) rather than all 30 values from 1 through 30. Picking 6 of these 16 values randomly with replacement yields $16^6 = 16\,777\,216$ possible arrangements of factors that could yield a product leaving a remainder of 4 when divided by 30. The only way to get an odd remainder is if all 6 picked values are odd, which has a probability of $\frac{1}{2^6} = \frac{1}{64}$, so $\frac{16{,}777{,}216}{64} = 262{,}144$ cases can be thrown out, leaving 16 515 072 potential patterns leaving the desired remainder of 4 upon division by 30. However, it is equally likely that a product of 6 values taken (repetitions allowed) from our 16, with at least 1 of the 6 being even, will yield any of our even options—2, 4, 8, 14, 16, 22, 26, 28 as the remainder when dividing the product of the 6 values by 30. Therefore, $\frac{1}{8}$ of the 16,515,072 candidates for a total of 2,064,384 will yield the actual remainder of 4, out of $30^6 = 729{,}000{,}000$ total sequences of numbers that can be multiplied. This yields a probability of randomly picking 6 integers from 1 to 30 (or, equivalently, 1 through 900) yielding a product that leaves a remainder of 4 when divided by 30 of $\frac{2{,}064{,}384}{729{,}000{,}000} = 0.0028318 \ldots \times 100\,\% = \mathbf{0.283}\%$ when rounded to the nearest 0.001 %.