2020 State Target Round Solutions

1. Joel works 8 hours per day and gets to keep 80% of his \$7.25/h wages. Therefore, to earn and keep \$2400 requires Joel to work: $\frac{\$2400}{8\frac{\text{h}}{\text{d}} \times 0.8 \times \$7.25/\text{h}} = \frac{\$2400}{\$46.40/\text{d}} = 51.72\ldots$ d, which rounds to **52** days.

2. We have already been given that the output of machine 2 is 3, so we pick up from there. For machine 3, the x input is from the original upper input, 5, and the y input is the output of machine 2, so 3, and we are to divide the x value by the y value to result in an output of $\frac{5}{3}$. For machine 4, the x input is output of machine 3, so $\frac{5}{3}$, and the y input is the output of machine 2, so 3, and we are to add the two inputs and then divide by 2 to result in an output of $\frac{7}{3}$. For machine 5, the x input is from the original upper input, 5, and the y input is the output of machine 4, so $\frac{7}{3}$, and we are to divide the x value by the y value to result in an output of $\frac{15}{7}$. For machine 6, the x input is output of machine 5, so $\frac{15}{7}$, and the y input is the output of machine 4, so $\frac{7}{3}$, and we are to add the two inputs and then divide by 2 to result in an output of $\frac{47}{21}$, which is equal to the mixed number $\mathbf{2\frac{5}{21}}$.

3. We need a fraction whose value in standard decimal form is between 0.60 and $0.6\overline{6}$. Sixths don't work because $\frac{3}{6} = 0.5$ is too small and $\frac{4}{6} = \frac{2}{3}$ is too large. Sevenths don't work since $\frac{4}{7} = 0.571\ldots$ is too small and $\frac{5}{7} = 0.714\ldots$ is too large. For eighths, $\mathbf{\frac{5}{8}} = 0.625$ works.

4. $S_k = 1 + 2 + 3 + \cdots + (k-1) + k = \frac{k(k+1)}{2}$, the sum of the first k positive integers, but this quantity for k iterating from n through $2n-1$ is always paired with a factor 2, as $2S_k = k(k+1)$, in P_n. Therefore, $P_n = [n(n+1)][(n+1)(n+2)][(n+2)(n+3)]\ldots[(2n-2)(2n-1)][(2n-1)(2n)]$, in which we a matching pair of factors in the right factor of one bracket and the left factor of the next bracket, for every bracket except the last. Therefore, $(n+1)$ through $(2n-1)$ all occur as squares that can be simplified out of the radical. The only remaining factors are the left factor of the first bracket and the right factor of the last bracket: $n(2n) = 2n^2$, whose square root is $n\sqrt{2}$. Upon simplification 2 is the only portion left in the radical, so $b = \mathbf{2}$.

5. The truth-tellers will correctly state how many truth-tellers there are, whereas the liars will state a wrong count of truth-tellers. The group of truth-tellers will be that group whose answer regarding the number of truth-tellers matches the number of people in the group. The only such match is 11, so the group claiming 11 truth-tellers are truth-tellers and they told the truth when they said there are **61** liars.

6. We work backwards. There's only 1 way to get to ◯ from ◯-you're already there! And there's 1 way to get to ◯ from each of the spaces adjacent to ◯. We place a blue 1 in each cell to show that there's 1 way to get to ◯ from the corresponding cell:

		Y	1	Φ
			X	1
				Z
⋆				

Next, consider the square with the X. From there, we can take one step right (and then there is 1 way to finish). Or we can take one step up (and then there is 1 way to finish). Combining these, we have 1 + 1 = 2 ways to finish from X.

Then, consider the square marked *Y*. From there, we can take one step right (1 way to finish from there), or two steps right (1 way to finish from there-already done!). Combining these, we have 1 + 1 = 2 ways to finish from *Y*. The situation at *Z* is identical to that at *Y*, except that we are going up instead of to the right. We'll keep exploiting this symmetry throughout this solution! Now, we have

	A	2	1	①
		B	2	1
			C	2
				D
★				

From *A*, we can go one step right (2 ways to finish), two steps right (1 way to finish), or three steps right (1 way to finish). So, there are 2 + 1 + 1 = 4 ways to finish from *A*. As with *Y* and *Z* above, we use symmetry to see that there are 4 ways to finish from Das well.

From *B*, we can go one step up (2 ways to finish), one step right (2 ways to finish), or two steps right (1 way to finish). So, there are 2 + 2 + 1 = 5 ways to finish from *B*. And a corresponding 5 ways to finish from *C*.

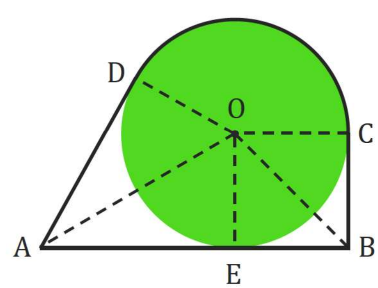
	4	2	1	①
		5	2	1
			5	2
				4
★				

We keep going in this manner. At each square as we work backwards, we find the number of paths from that square by adding all the numbers in the squares directly to the right of that square and all the numbers directly above that square. We use symmetry as above to cut down on the calculations, and we finally find:

8	4	2	1	①
28	12	5	2	1
94	37	14	5	2
289	106	37	12	4
838	289	94	28	8

We see that the number sequences of moves to get from ★ to ◯ is **838** sequences.

7. Let's draw in the figure some more auxiliary lines connecting the center of circle O points of tangency of line segments to O. A radius to a point of tangency is perpendicular to the tangent line, so $\overline{OC} \perp \overline{CB}$, $\overline{OE} \perp \overline{AB}$, and $\overline{OD} \perp \overline{DA}$. Thus, △OBC is a right triangle. The hypotenuse OB is given as $10\sqrt{2}$ cm and radius OC is given as 10 cm, making BC to be 10 cm and △OBC a right isosceles triangle. Similar analysis of △OBC shows BE to be 10 cm as well. With 4 congruent sides and some right angles, we know that OCBE is a square, making the measures of ∠OEC and arc EC be 90 degrees. Triangles OAD and OAE are right triangles with legs OD and OE, respectively, being half the hypotenuse OA, so the two triangle are 30-60-90, with angles DOA and AOE measuring 60 degrees connecting the respective short leg and the hypotenuse, so the combined angle DOE and corresponding arc DE measuring 120 degrees. When we combine arcs DE and EC into major arc DEC, the resulting measure is 210 degrees.



This is where the rope is *not* in contact with the watermelon. The rope is in contact with the watermelon over the remaining part of the circle O, which is the minor arc DC, which has measure $360 - 210 = 150$ degrees, for $\frac{5}{12}$ of the circumference of O, making an arc length of $\frac{5}{12} \times 2\pi(10) = \frac{25}{3}\pi$ cm. Segment DA is the long leg of 30-60-90 $\triangle AOD$, and thus has length $\sqrt{3}$ times as long as the 10 cm of the short leg OD for $10\sqrt{3}$ cm. Therefore, the total length of the rope is $\left(10 + \frac{25}{3}\pi + 10\sqrt{3}\right)$ cm $= 53.5004\ldots$ cm, which rounds to the nearest 0.1 cm as **53.5** cm.

8. We have information about the powers that start with 8, and we want to learn about the powers that start with 9. So, we think about how these powers get produced in Tyrell's list. Each power is double the power before it in the list. We get a number that starts with 8 or 9 when we double a number that starts with 4. That's promising-we found something in common between the powers we know something about (the ones that start with 8) and the ones we want to know about (the ones that start with 9).

We can only get a power that starts with 4 if we are doubling a number that starts with 2. But we don't always get a power that starts with 4 when we double a number that starts with 2! Sometimes we get a power that starts with 5. Hmmm ... And we can only get a power that starts with 2 if we're doubling a power that starts with 1. But sometimes doubling a power that starts with 1 will give us a power that starts with 3.

That's enough working backwards. Now that we have some idea of how we can get a power that starts with 8 or 9 (we need a power that starts with 4), let's take a look at the first several powers of 2 to see if we find anything interesting. We'll group the powers by the number of digits the powers have, since we have a little information now about the patterns we expect as we double numbers starting with a number that begins with 1.

1 digit: 1,2,4,8
2 digits: 16, 32, 64
3 digits: 128,256,512
4 digits: 1024, 2048, 4096, 8192

Right away, we see some of the patterns that we expected. We only have a number that starts with 8 or 9 in groups that have a power that starts with 4. We also see that each group has a power that starts with 1. That's because any k-digit power that starts with a number greater than 1 is double another k-digit power. So, there must be a power that starts with 1 in each group.

Doubling a number that starts with 1 gives a number that starts with 2 or 3, so the second power in each group starts with 2 or 3.

Doubling a number that starts with 2 or 3 gives a number that starts with 4, 5, 6 or 7. So, the third power in each group starts with one of these four digits.

If the third power in the group starts with 4, then there will be a fourth power in the group, and that power will start with 8 or 9. Otherwise, the group only has three powers.

Aha! We now know that every group has three or four powers, and that the powers starting 8 or 9 must the be the fourth powers in the four-power groups. We have enough information to tackle the problem now!

Tyrell's list has the first digits of 1001 powers. Because the last power has 302 digits, our group-by-digit approach above produces 302 groups. From the given information about the end of Tyrell's list, we know that the last group has 1 power. The rest of the groups have 3 or 4 powers. Removing that last group, we have 301 groups that together have 1000 powers. Removing the first 3 powers of each group (so that many end up empty), we take away $3 \cdot 301 = 903$ powers, leaving $1000 - 903 = 97$ powers, each of which is the last power in a group of 4 powers. That is, each of these remaining 97 powers starts with 8 or 9. We are told that 52 of these powers start with 8, so the other $97 - 52 = 45$ start with 9. So, 9 appears **45** times.