Division **M**

Mathematical Olympiads

November 2023

for Elementary & Middle Schools

Contest **1**

Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ______________________________

1A How many integers are greater than $(-2)^3$ but less than 3^2?

1B The number 2023 can be factored into $a \times b^2$. Given that a and b are prime numbers and $a + b = 24$, find $a - b$.

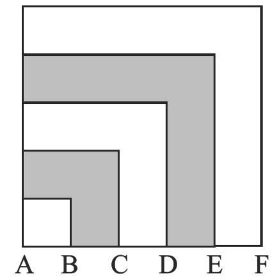
1C A recent survey of mathematicians and chess players found that 50% of mathematicians surveyed are chess players, and 20% of chess players surveyed are mathematicians. If N% is the percent of those surveyed who are both mathematicians and chess players, find N to the nearest whole number.

Name: ______________________________

Answer Column
1A
1B
1C
1D
1E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

– Page may be folded along dotted line. –

1D Five squares are nested so that their lower left vertices are at A. Points A, B, C, D, E, and F are collinear such that AB = BC = CD = DE = EF, as shown. Let S = the sum of the areas of the shaded regions and let U = the sum of the areas of the unshaded regions. Compute the ratio S/U and express your answer as a fraction in simplest form.



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1E Forgetful Freddie attempted to sum the whole numbers from 1 to 30 inclusive. He mistakenly left out one of the numbers and got a sum which was 1 less than a perfect square. Determine the whole number that Forgetful Freddie mistakenly omitted.

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SOLUTIONS AND ANSWERS

1A **METHOD 1** *Strategy: List the possible integers.*
Notice that $(-2)^3 = (-2)(-2)(-2) = -8$ and $(3)^2 = (3)(3) = 9$. The integers between these values are −7, −6, −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, 6, 7, 8, which means there are **16** integers.

METHOD 2 *Strategy: Count the number of integers using subtraction.*
Apply the information from the first sentence in method 1. The largest integer less than 9 is 8, and the smallest integer greater than −8 is −7. Subtraction counts the number of one-unit intervals between two numbers: $8 - (-7) = 15$. To find the number of integers add 1 to the number of intervals: $15 + 1 = 16$.

FOLLOW UPS: (1) How many even integers are between −100 and 100, inclusive? [101] *(2) How many integers are greater than 5^2 but less than 2^5?* [6]

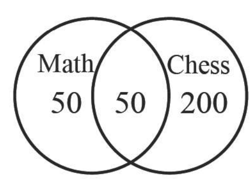
1B **METHOD 1** *Strategy: List pairs of primes that add to 24.*
Since 2023 is odd, it has no even factors. List pairs of prime numbers whose sum is 24: (5, 19), (7, 17), and (11, 13). Reject (5, 19) since 5 is not a factor of 2023. Reject (11, 13) by applying the test for divisibility by 11: $2 - 0 + 2 - 3 = 1$, which is not a multiple of 11. Thus, $2023 = 7 \times 17^2$ and $7 - 17 =$ **–10**.

METHOD 2 *Strategy: Consider each prime in numerical order to find a and b.*
Since 2023 is odd, 2 is not one of its factors. Since $2 + 0 + 2 + 3 = 7$, 3 is not a factor of 2023. Since 2023 does not have either 0 or 5 in the ones position, 5 is not a factor either. If $a = 7$, then $b = 24 - 7 = 17$. Verify that $7 \times 17 \times 17 = 2023$. Therefore, $a = 7$ and $b = 17$. It follows that $7 - 17 = -10$.

FOLLOW UP: Find the sum of the prime factors of 2024. [36]

1C **METHOD 1** *Strategy: Use a numerical approach.*
Since 1/2 the mathematicians do both, assume there are 2 mathematicians, which means 1 of them is only a mathematician, and 1 person does both. If 1 person does both, and that 1 person is 20% of all chess players, that means there are 5 chess players, 4 of whom only play chess. This means there are $1 + 1 + 4 = 6$ total people, so 1/6 = 16 2/3% of everyone surveyed are both mathematicians and chess players. Thus, N = 16 2/3 rounded to **17**.

METHOD 2 *Strategy: Draw a Venn Diagram.*
Choose a "convenient" number for the number of mathematicians, say 100. Use the idea of percent to complete the diagram. Since 20% of the chess players are mathematicians, the number of "just" chess players needs to be 200. The required fraction becomes: $\dfrac{50}{50+50+200} = \dfrac{50}{300} = \dfrac{1}{6} = 16\tfrac{2}{3}\%$. This rounds to 17%.

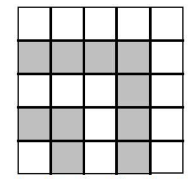


1A	**16**
1B	**–10**
1C	**17**
1D	**2/3**
1E	**25**

FOLLOW UP: In a survey of 200 students, 100 take geometry, 150 take chemistry, and 90 students take both geometry and chemistry. How many students take neither geometry nor chemistry? [40]

1D **METHOD 1** <u>*Strategy*</u>*: Add the areas of all the shaded shapes and then add the areas of the rest.*
Let AB = 1. The sum of the shaded areas is $S = (2^2 – 1^2) + (4^2 – 3^2) = 3 + 7 = 10$.
The sum of the unshaded areas is $U = 1^2 + (3^2 – 2^2) + (5^2 – 4^2) = 1 + 5 + 9 = 15$.
It follows that S/U = 10/15 = **2/3**.

METHOD 2 <u>*Strategy*</u>*: Draw vertical and horizontal lines to create a 5 × 5 grid.*
Draw the lines and count the shaded and unshaded squares. There are 10 shaded squares and 15 unshaded squares. Thus, S/U = 10/15 = 2/3.



FOLLOW UPS: (1) If another shaded region were added in the same pattern so that FG = AB, etc., what would be the ratio of the area of the shaded region to the area of the unshaded region? Express your answer as a fraction in simplest form. [7/5] *(2) Examine the sequence of ratios S/U when an odd number of squares are nested (i.e., largest square is unshaded.) What will S/U be if 15 squares are nested?* [7/8]. *(3) If S/U = 23/24, how many nested squares are there?* [47]

1E **METHOD 1** <u>*Strategy*</u>*: Find the range of values for Freddie's sum.*
Apply Gauss' formula to find S = (1 + 30) × 30/2 = 465. Freddie's sum must be greater than or equal to 465 – 30 = 435 and less than or equal to 465 – 1 = 464. The only perfect square between these two numbers is $21^2 = 441$ and 1 less than 441 is 440. Since 465 – 440 = 25, the omitted number was **25**.

METHOD 2 <u>*Strategy*</u>*: Look at the perfect squares near 465.*
Since the sum of the whole numbers from 1 to 30 equals 465 (see method 1), consider perfect squares near 465. Consider $20^2 = 400$, $21^2 = 441$, and $22^2 = 484$. Since 484 is more than 465, Freddie either got 400 – 1 = 399 or 441 – 1 = 440. The difference between 399 and 465 is 66 which is not a number between 1 and 30. The difference between 440 and 465 is 25. Thus, Freddie left out 25 and got a sum of 440.

FOLLOW UP: Forgetful Freddie attempted to sum the whole numbers from 1 to 10 inclusive. He mistakenly left out two of the numbers and got a sum that is a prime number. What is the greatest possible sum for the two numbers Freddie omitted? [18]