Division **M** | Mathematical Olympiads | November 2025 | for Elementary & Middle Schools | Contest **1**

SOLUTIONS AND ANSWERS

Answer Column
1A 2025
1B 22
1C 5^{150}
1D 36
1E 92

1A **METHOD 1** *Strategy: Substitute and use the rules for order of operations.*
Find N: $N = 2^1 + 2^3 + 2^5 = 2 + 8 + 32 = 42$. Then $(42 + 3)^2 = 45^2 = \mathbf{2025}$.

METHOD 2 *Strategy: Square the binomial and then substitute.*
Square the given expression: $(N + 3)^2 = N^2 + 6N + 9$. Calculate N as in METHOD 1 and then substitute to get $42^2 + 6(42) + 9 = 1764 + 252 + 9 = 2025$.

FOLLOW UP: Two consecutive perfect squares differ by 85. Find the value of the smaller perfect square. [1764]

1B **METHOD 1** *Strategy: Find the number of lattice points for each integer y-value.*
The slope of the hypotenuse is 1/3, which means that when y decreases by 1, x decreases by 3, placing 3 additional lattice points in or on the triangle.
When $y = 3$ there is 1 point. When $y = 2$ there are 1 + 3 = 4 points. When $y = 1$ there are 4 + 3 = 7 points. When $y = 0$ there are 7 + 3 = 10 points.
The total is 1 + 4 + 7 + 10 = **22** points.

METHOD 2 *Strategy: Count the number of coordinates by making a list.*
The coordinates of the lattice points that are contained inside or on the right triangle are (–3, 0), (–2, 0), (–1, 0), (0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (3, 2), (4, 2), (5, 2), (6, 2), (6, 3). There are 22 sets of coordinates.

METHOD 3 *Strategy: Apply Pick's theorem.*
Pick's theorem expresses the area of a polygon, all of whose vertices are lattice points in a coordinate plane, in terms of the number of lattice points inside the polygon (I) and on the polygon (B). The formula is: $A = I + \frac{B}{2} - 1$. Count the boundary points: 10 on the triangle's horizontal leg and an *additional* 3 on the vertical leg. Since the hypotenuse has slope 1/3, lattice points lie on it at $x = -3$, 0, 3, and 6. Two of these have already been counted; the total number of boundary points is 10 + 3 + 2 = 15. The area of the triangle is $(3 \times 9)/2 = 27/2$. Substitute into Pick's equation: $27/2 = I + 15/2 - 1$ so $I = 7$. Therefore, the total number of lattice points contained inside or on the right triangle is 15 + 7 = 22.

FOLLOW UP: How many lattice points are on the circumference of a circle that is centered at the origin with a radius of 5? [12]

1C *Strategy: Recognize that each exponent is a multiple of 50.*
Since each exponent is a multiple of 50, rewrite each number as follows: $2^{250} = (2^5)^{50}$, $3^{200} = (3^4)^{50}$, $5^{150} = (5^3)^{50}$, and $7^{100} = (7^2)^{50}$. Compare the bases: $2^5 = 32$, $3^4 = 81$, $5^3 = 125$, and $7^2 = 49$. Thus, $\mathbf{5^{150}}$ has the greatest value.

FOLLOW UP: Which has the least value of 9^{72}, 6^{54}, or 11^{27}? [11^{27}]

1D **METHOD 1** <u>Strategy</u>: *Use permutations.*

Find the total number of arrangements. If all 5 letters were different, there are 5! = 5 × 4 × 3 × 2 × 1 = 120 ways to rearrange the letters. Since the word FUZZY consists of 5 letters: F, U, Z, Z, Y, where the letter Z appears twice, we divide 120 by 2 to get 60 arrangements. Treat ZZ as a single unit, and arrange the four elements: F, U, ZZ, Y. The number of ways to arrange these four units is: 4! = 24. To find the number of arrangements without ZZ, subtract the cases where they are together from the total: 60 – 24 = **36**.

METHOD 2 <u>Strategy</u>: *Create a systemized list.*

List the arrangements that begin with the letter F, but does not contain ZZ: FUZYZ, FYZUZ, FZYUZ, FZYZU, FZUYZ, FZUZY. Similarly, there will be 6 arrangements beginning with U and 6 beginning with Y. There are 6 arrangements beginning with ZF, 6 beginning with ZU, and 6 beginning with ZY. The total number of arrangements is 6 × 6 = 36.

FOLLOW UP: How many ways can the letters of the word BANANA be arranged so that no identical letters are next to each other? [10]

1E **METHOD 1** <u>Strategy</u>: *Look for a pattern and then generalize.*

Count the number of squares in each element of the sequence and then determine the differences between consecutive terms. The 12th term equals the first term, 4, plus the sum of the next 11 differences beginning with 3 and ending with 13. Therefore, the 12th term is 4 + (3 + 4 + 5 + … + 12 + 13). There are several ways to find the sum of the numbers in the parentheses. For example: Sum = (first + last) × (number of terms)/2 = (3 + 13) × 11/2 = 88. The number of squares in the 12th picture in the sequence is 4 + 88 = **92**.

4   7   11   16   22   …

 3   4   5   6   7   …

METHOD 2 <u>Strategy</u>: *Create a table.*

Term (n)	1	2	3	4	5	6	7	8	9	10	11	12
Amount added		3	4	5	6	7	8	9	10	11	12	13
Total	4	7	11	16	22	29	37	46	56	67	79	92

Thus, there are 92 squares of size 1 × 1 in the 12th figure.

METHOD 3 <u>Strategy</u>: *Create an algebraic formula to generate the terms.*

Let n = the position of the term in the sequence and S_n = the sum of the first n terms. Since a_n grows more quickly than the values of n, consider using n^2 in the formula. Using finite differences, generate the sequence $S_n = \frac{1}{2}n^2 + \frac{3}{2}n + 2 = \frac{n(n+3)}{2} + 2$.

A recursive formula may also be found. This type of formula finds a_n in terms of previous terms and requires defining the first term. In this case, $a_1 = 4$ and $a_n = a_{n-1} + (n + 1)$. To find a_{12} one must find a_{11} since $a_{12} = a_{11} + (12 + 1)$. Similarly, to find a_{11} one must find a_{10} since $a_{11} = a_{10} + (11 + 1)$. This process is exactly what was done in METHOD 2 and is generally not as efficient to use as the closed form of the equation.

FOLLOW UP: What is the total perimeter of the 12th term in the "spiral worm" sequence? [186]