Triangles Stretch

21. Point T divides side QR into segments QT and RT, so that QR = QT + RT = 26 cm. We are told that the ratio of QT to RT is 9 to 4. That means QT = (9/13)QR = (9/13)(26) = 18 cm, and RT = 26 − 18 = 8 cm. By the geometric mean theorem, then, altitude ST = $\sqrt{(QT \times RT)} = \sqrt{(18 \times 8)} = \sqrt{144}$ = **12** cm.

22. The altitude of isosceles triangle WXY drawn perpendicular to side XY intersects the side at its midpoint, creating two congruent right triangles, each with a leg of length 8/2 = 4 cm and a hypotenuse of length 5 cm. The second leg of each right triangle is the altitude in question, with length *h*. We can use the Pythagorean theorem to determine *h* as follows: $4^2 + h^2 = 5^2 \rightarrow 16 + h^2 = 25 \rightarrow h^2 = 9 \rightarrow h = \sqrt{9} = \mathbf{3}$ cm. Alternatively, you might recognize that these are 3-4-5 right triangles and deduce that the length of the altitude must be **3** cm.

23. The altitude drawn perpendicular to the 10-cm side divides the isosceles triangle into congruent right triangles, each with a leg of length 10/2 = 5 cm and a hypotenuse of length 13 cm. We can use the Pythagorean theorem to determine that this altitude, which is a shared leg of the two right triangles, has length $\sqrt{(13^2 - 5^2)} = \sqrt{(169 - 25)} = \sqrt{144} = 12$ cm. You might also recognize that these are 5-12-13 right triangles and deduce that the length of this altitude must be 12 cm. With a base of length 10 cm and height 12 cm, the triangle has area (1/2)(10)(12) = 60 cm^2. When the base is a side of length 13 cm, the triangle has height *h*. Since the triangle has area 60 cm^2, we have 60 = (1/2)(13)*h*. So, 13*h* = 120 and *h* = 120/13 cm. There are two sides of length 13 cm, so it follows that the three altitudes of this triangle have lengths 12 cm, 120/13 cm and 120/13 cm. The average of their lengths is [(120/13) + (120/13) + 12] ÷ 3 = [(240 + 156)/13] ÷ 3 = (396/13) ÷ 3 = (396/13) × (1/3) = **132/13** cm.

24. The base of triangle PQR is side PR, and median QS divides base PR into congruent segments PS and RS. Since PS = 9 cm, it follows that RS = 9 cm and PR = 9 + 9 = 18 cm. We are told that triangle PQR has area 24 cm^2. Substituting these values into the area formula $A = (1/2)bh$, we can determine the height of triangle PQR. Doing so, we get 24 = (1/2)(18)*h*. So, 9*h* = 24, and *h* = 24/9 = **8/3** cm.

25. By the centroid theorem, we know that 2DG = AG. Substituting AG = 6*x* and DG = 5*x* − 8 into this equation, we get 2(5*x* − 8) = 6*x*. Now solving for *x* gives us $10x - 16 = 6x \rightarrow 4x = 16 \rightarrow x = \mathbf{4}$. P

26. We are told that NP = 5*x*/2 + 2 and KN = 19*x*/2. By the centroid theorem, we know that 2NP = KP. So, KP = 2(5*x*/2 + 2) = 5*x* + 4. Given that KN = KP + NP, it follows that KP = KN − NP = 19*x*/2 − (5*x*/2 + 2) = 19*x*/2 − 5*x*/2 − 2 = 7*x* − 2. Now we have two expressions for KP that we can set equal to each other to get 5*x* + 4 = 7*x* − 2. Solving for *x*, we get 2*x* = 6, and *x* = 3. Therefore, KP = 5(3) + 4 = 15 + 4 = **19** units. Substituting into the other expression, we can confirm that KP = 7(3) − 2 = 21 − 2 = **19** units.

27. The triangle formed by connecting the midpoints of the sides of a triangle, as is the case with triangle HIJ, is called a medial triangle. The ratios of corresponding sides of triangles HIJ and EFG are all 1:2, so these triangles are similar, with HI = (1/2)EF = (1/2)(25) = 25/2 cm, IJ = (1/2)FG = (1/2)(17) = 17/2 cm, and HJ = (1/2)EG = (1/2)(26) = 13 cm. We can determine the area of triangle HIJ by using Heron's formula, which states that a triangle with side lengths *a*, *b* and *c* and semiperimeter *s* has area $\sqrt{[s(s-a)(s-b)(s-c)]}$. For triangle HIJ, *s* = (25/2 + 17/2 + 13)/2 = (21 + 13)/2 = 17 cm. The area, then, is $\sqrt{[17(17 - 25/2)(17 - 17/2)(17 - 13)]} = \sqrt{[17((34 - 25)/2)((34 - 17)/2)(4)]} = \sqrt{[17(9/2)(17/2)(4)]} = \sqrt{[17(9)(17)]} = 17 \times 3 = \mathbf{51}$ cm^2. Alternatively, given that corresponding sides of triangles HIJ and EFG are all in the proportion 1:2, it follows that the areas of triangles HIJ and EFG must be in the proportion $1^2/2^2 = 1/4$. That means the area of triangle HIJ is 1/4 the area of triangle EFG. Now we can determine the area of triangle EFG, using Heron's formula, for *a* = 25, *b* = 17, *c* = 26 and *s* = (25 + 17 + 26)/2 = 68/2 = 34 cm. Triangle EFG has area $\sqrt{[34(34 - 25)(34 - 26)(34 - 17)]} = \sqrt{[34(9)(8)(17)]} = \sqrt{(2^4 \times 3^2 \times 17^2)} = 2^2 \times 3 \times 17 = 204$ cm^2. Therefore, the area of triangle HIJ is 1/4 that, or 1/4(204) = **51** cm^2.

28. The perimeter of triangle PQR is PQ + QR + PR. We know that PQ = 16 cm. Also, QR = QS + RS, and we know RS = 11 cm and QS = 8 cm, so QR = 8 + 11 = 19 cm. Since segment PS is an angle bisector of triangle PQR, by the angle bisector theorem, we know that PQ/PR = QS/RS. Substituting the known values gives us 16/PR = 8/11. Solving for PR, we get PR = 16(11/8) = 2(11) = 22 cm. Therefore, triangle PQR has perimeter 16 + 19 + 22 = **57** cm.

29. Segment BE, which is an angle bisector of triangle ABC, is also an angle bisector of triangle ABD. By the angle bisector theorem, BD/AB = DG/AG. We know that AB = 10, BC = 7 and AC = 11, but we need to find BD. We know that AD is an angle bisector of triangle ABC, so by the angle bisector theorem, AB/AC = BD/CD, and we have 10/11 = BD/CD. Given that BC = BD + CD = 7, we now know that BD = (10/21)BC = (10/21)(7) = 10/3. So, the ratio of DG to AG is BD/AB = (10/3)/10 = **1/3**.

30. Since AB = AD + BD, and we are told that AB = 15 cm and BD = 6 cm, we can determine that AD = 15 − 6 = 9 cm. Additionally, segments from a single exterior point of a circle to two different points of tangency on the circle are congruent. Thus, BD = BE = 6 cm, AD = AF = 9 cm and CE = CF. And since BC = BE + CE, it follows that CE = 14 − 6 = 8 cm = CF. Finally, AC = AF + CF = 9 + 8 = **17** cm.