Division **M** Mathematical Olympiads for Elementary & Middle Schools

January 9, 2023

Contest **3**

Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: _____

3A If $\frac{a}{b} = -2$, compute the value of $a + 2b$.

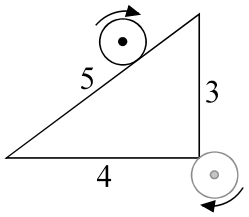
3B The formula for the area of a triangle is $A = bh/2$.
If the base increases by 10%, and the height increases by 10%, then the area A increases by $X\%$. Find the exact value of X.

3C For how many different displays, starting at 10:00 AM and ending at 4:00 PM, during a particular day, will a digital clock show that the number of the hour is less than the number of the minute? [For example, at 2:15 the number of the hour is less than the number of the minute but at 2:01, the number of the hour is greater than the number of the minute.]

Name: _____

Answer Column
3A
3B
3C
3D
3E
Do Not Write in this Space. *For PICO's Use Only.* *SCORE:*

3D A circle with radius 1-inch rolls without slipping along the sides of a right triangle whose side lengths are 3 inches, 4 inches, and 5 inches as shown. When the circle gets to a vertex, it continues to rotate around the vertex. The circle starts and ends at the middle of the 5-inch side. The center of the circle travels a total distance of $M + N\pi$ inches in one complete circuit. Find $M + N$ in simplest form.



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3E Express the quotient of repeating decimals $\frac{0.166\overline{6}}{0.\overline{5555}}$ as the ratio of two whole numbers with no common factor other than 1.

– Page may be folded along dotted line. –

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SOLUTIONS AND ANSWERS

3A **METHOD 1** *Strategy: Use algebra.*

Multiply each side of the given equation by b: $\frac{a}{b} = -2 \rightarrow \frac{a}{b} \times \frac{b}{1} = -2 \times b$ so $a = -2b$ and $a + 2b = -2b + 2b = \mathbf{0}$.

METHOD 2 *Strategy: Choose numbers that satisfy the equation.*
Let $a = -6$ and $b = 3$ so $a/b = -2$. Substitute values for a and b: $-6 + 2 \times 3 = 0$.

FOLLOW UP: Use $a/b = -2$ to find the value of $a^2 + 4ab + 4b^2$. [0]

3B **METHOD 1** *Strategy: Use the formula for the area of a triangle and compare.*
When the base increases by 10% the new base = $1.1b$. When the height increases by 10% the new height is $1.1h$. The area of the new triangle is $(1.1h) \times (1.1b)/2 = 1.21bh/2 = (1 + 0.21)bh/2$. Compare this to the original area to see that it is 21% greater. Thus, $X = \mathbf{21}$.

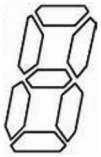
METHOD 2 *Strategy: Select convenient values for the base and height.*
Let the base of the triangle equal 20 and the height equal 10. The area of the triangle equals $(20 \times 10)/2 = 100$. When the base is increased by 10% it becomes $1.1 \times 20 = 22$ and when the height is increased by 10% it becomes $1.1 \times 10 = 11$. The area of the new triangle is $(22 \times 11)/2 = 121$. The area increased by 21 out of 100 or 21%.

FOLLOW UP: If the base of a triangle increases by 20% and the height decreases by 20%, what would be the percentage increase (decrease) of the area of the resulting triangle compared to the original triangle? [decrease by 4%]

3C **METHOD 1** *Strategy: Analyze the situation piecewise.*
Find the time intervals when the hour is less than the minutes: 10:11 to 10:59, 11:12 to 11:59, 12:13 to 12:59, 1:02 to 1:59, 2:03 to 2:59, and 3:04 to 3:59. Sum the number of occurrences in each interval: $49 + 48 + 47 + 58 + 57 + 56 = \mathbf{315}$.

METHOD 2 *Strategy: Subtract the number of minutes $H \geq M$ from the total time.*
The number of minutes from 10:00 to 4:00 is $6 \times 60 = 360$ minutes. The hour is greater than or equal to the minute from: 10:00 to 10:10, 11:00 to 11:11, 12:00 to 12:12, 1:00 to 1:01, 2:00 to 2:02, and 3:00 to 3:03 for a total time of $11 + 12 + 13 + 2 + 3 + 4$ equals 45 minutes. Subtract that from 360 minutes to get $360 - 45 = 315$ occurrences.

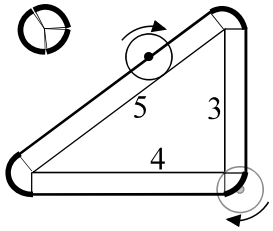
FOLLOW UPS: (1) Between noon and midnight on the same day, the minute number displayed on a digital clock is a factor of the hour number displayed K times. Find K. [35] *(2) Each digit on a digital alarm clock is illuminated by a 7-segment LED display, as shown at the right. At times between 9:00 AM and 5:00 PM, K segments are illuminated. Determine the maximum possible value of K, and all times when K segments are illuminated.* [21; occurs once at 10:08]



3A	3B	3C	3D	3E
0	21	315	14	3/10

3D *Strategy: Use the formula that circumference = 2πr.*

Notice that the path that the center of the circle traverses is comprised of 3 segments parallel to and equal in length to each of the three sides. At each of the 3 vertices it travels along arcs. The sum of the 3 arcs is one complete revolution of the circle or 360° (see diagram). The circumference = 2π(1) = 2π. The total distance traveled is 3 + 4 + 5 + *2*π = 12 + *2*π inches. Therefore, *M* = 12 and *N* = 2, so *M* + *N* = 12 + 2 = **14**.



FOLLOW UP: Suppose the same circle rolls without slipping along the 4 sides of a square whose area is 25 sq. in. What is the total distance traveled by the center in one complete circuit? [20 + 2π inches]

3E **METHOD 1** *Strategy: Express the numerator and denominator in fraction form.*

Convert the repeating decimal $0.166\overline{6}$ into a fraction as follows: let $n = 0.166\overline{6}$, then $10n = 1.66\overline{6}$. Subtract the equations to get $9n = 1.5$ so $90n = 15$ and $n = 1/6$. Similarly, let $m = 0.555\overline{5}$, then $10m = 5.55\overline{5}$. Subtract the equations to get $9m = 5$ so $m = 5/9$. Thus, $\dfrac{0.166\overline{6}}{0.555\overline{5}} = \dfrac{1/6}{5/9} = \dfrac{1}{6} \times \dfrac{9}{5} = \mathbf{\dfrac{3}{10}}$.

METHOD 2 *Strategy: Use a form of long division.*

Since the numerator is less than the denominator, the quotient is less than 1. Multiply numerator and denominator by 10000 and ignore the repeat sign. Divide the denominator into the numerator to get an approximate result. Notice that $0.3 \times 5555 = 1666.5$. If the number of 6's and 5's in the two numbers were each increased by the same number of 6's and 5's, the quotient would still be 0.3 = 3/10. That means if we multiply $0.555\overline{5}$ by 0.3 the result will be $0.166\overline{6}$.

$$\begin{array}{r} 0.3 \\ 5555 \overline{)\,1666.6} \\ \underline{1666.5} \end{array}$$

FOLLOW UPS: (1) Express 22/7 as a repeating decimal. [$3.\overline{142857}$]

(2) Determine the decimal value for: (a) $\dfrac{0.\overline{72} + 0.\overline{27}}{0.\overline{72} - 0.\overline{27}}$ *and (b)* $\left(0.\overline{324}\right) \times \left(1.541\overline{6}\right)$ [11/5 = 2.2 and 0.5]

(3) Rewrite the number $\dfrac{1}{0.\overline{324}}$ *as both a repeating decimal and a simplified fraction.* [$3.08\overline{3}$ and 37/12]

NOTE: *Other FOLLOW UP problems related to some of the above can be found in our four contest problem books and in "Creative Problem Solving in School Mathematics." Visit www.moems.org for details and to order.*