Workout 2

151. There would be 8! = 40,320 different arrangements if all the letters of MATHLETE were different, but the word has two T's and two E's, so there are only 40,320 ÷ 4 = 10,080 unique arrangements of the eight letters. To find out how many of these arrangements have the two T's next to each other, we can pretend that we have TT glued together as a single item. There are then 7!/2 = 2520 arrangements with TT. Likewise, there are 2520 arrangements with EE. We should note that we have double counted the arrangements that have both TT and EE. There are 6! = 720 of those. To find the number of arrangements without TT or EE, we calculate 10,080 − 2520 − 2520 + 720 = 5760, adding back in the arrangements that we subtracted twice. The probability that a random arrangement of the letters has no two consecutive letters is thus 5760/10,080 = **4/7**.

152. The five terms of the first sequence are 6, 11, 16, 21, 26. The five terms of the second sequence must be 19, 14, 9, 4, −1. The absolute difference between the first and last terms of the second sequence is |19 − (−1)| = **20**.

153. The first 10 positive three-digit integers with distinct digits are 102, 103, 104, 105, 106, 107, 108, 109, 120 and 123. Their sum is **1087**.

154. Since the radius is 6 cm, the diameter—and thus the diagonal of the square—is 12 cm. The side length of a square is its diagonal divided by $\sqrt{2}$, so the side length is $12 \div \sqrt{2} \approx$ **8.5** cm.

155. A number with 16 factors can have a prime factorization of the form p^{15}, $p^7 \times q$, $p^3 \times q^3$, $p^3 \times q \times r$, or $p \times q \times r \times s$. The least number of the first form is 2^{15} = 32768, which has 15/16 of its factors even. The least number of the second form is $2^7 \times 3 = 128 \times 3 = 384$, with 14/16 = 7/8 of its factors even. The least number of the third form is $2^3 \times 3^3 = 8 \times 27 = 216$, with 12/16 = 3/4 of its factors even, making it a candidate for the least such number. The least number of the fourth form is $2^3 \times 3 \times 5 = 120$, also with 12/16 = 3/4 of its factors even, so this is our least candidate so far. The least number of the last form is $2 \times 3 \times 5 \times 7 = 210$, with 8/16 = 1/2 of its factors even. Thus, the number N must be **120**.

156. The only way $f(x)$ can be an integer is if the value of the denominator, $2x - 37$, is a divisor of the numerator, 237. The divisors of 237 are 1, 3, 79, 237 and their opposites, −1, −3, −79, −237. Solving an equation for each of these possible denominators, we get the corresponding x-values 19, 20, 58, 137 and 18, 17, −21, −100. The sum of these values is 269 − 121 = **148**.

157. The first 17 positive integers have remainders of 1, 2, 3, … 16 and 0 when divided by 17. The sum of these first 17 remainders is 16 × 17 ÷ 2 = 136. If we were continuing until 102, this cycle would repeat 6 times and the total would be 6 × 136 = 816. Since we are summing the remainders only to 100, we need to subtract from 816 the last two remainders of 16 and 0. The answer is 816 − 16 − 0 = **800**.

158. The square root of 2025 is 45, so the product of the side lengths of our triangle should not exceed 45. At the low end, we can assign the least positive integer 1 to the shortest side of this 30-60-90 triangle. The side length, in this case, would be 1, $\sqrt{3}$ and 2 and their product would be $2\sqrt{3}$. The square of this product would be $S_1 = (2\sqrt{3})^2 = 12$. The next set of possible side lengths is 2, $2\sqrt{3}$ and 4, with a product of $16\sqrt{3}$ and a square $S_2 = (16\sqrt{3})^2 = 768$. Then we have 3, $3\sqrt{3}$ and 6, with a product of $54\sqrt{3}$ and a square $S_3 = (54\sqrt{3})^2 = 8748$, which is too big. The sum of the possible values of S is 12 + 768 = **780**.

159. Justin can use an even number of quarters from 0 to 20 and complete the rest of the $5.00 with dimes. Including zero, that's **11** even numbers, so there are **11** combinations of quarters and dimes he can use to pay $5.00 in the board game.

160. The prime factorization of 24 is $2^3 \times 3$. To make a perfect square, we need an even number of each prime factor, so we just need to multiply 24 by 2 and 3. The answer is 24 × 6 = **144**.